A Note on Hack’s Conjecture, Parikh Images of Matrix Languages and Multiset Grammars

Georg Zetzsche

University of Hamburg, Department of Computer Science
georg.zetzsche@informatik.uni-hamburg.de

Abstract

It is shown that Hack’s Conjecture on Petri nets implies that for every language generated by a matrix grammar (without appearance checking), there is a non-erasing matrix grammar generating a language of the same Parikh image. Is is also shown that in this case, the classes of multiset languages generated by arbitrary and monotone multiset grammars coincide.

1 Introduction

For definitions and notation, we refer the reader to [Zet09].

In [Hac76, p. 172], Michel Hack conjectured that the reachability problem for Petri nets is decidable. The conjecture also states that this is due to the fact that for every Petri net of size $y \in \mathbb{N}$, a constant $K_y \in \mathbb{N}$ can be determined such that for any marking $\mu$, the zero marking is reachable from $\mu$ iff it can be reached by a firing sequence of length less than $K_y \cdot ||\mu||$.

\footnote{Hack defines the size of a Petri net to be its total number of arcs (see [Hac76, p. 170]).}
Although it is known that the reachability problem for P/T nets is decidable (see [May81, May84]), this stronger conjecture remains open and has interesting implications. In this paper, it is shown that if Hack’s Conjecture holds, the classes MAT and MAT$^\lambda$ coincide with respect to their Parikh image. That is, for every matrix grammar $G$ (without appearance checking), there is a non-erasing matrix grammar $G'$ such that $\Psi(L(G')) = \Psi(L(G))$. In other words, $\Psi(MAT^\lambda) \subseteq \Psi(MAT)$ and therefore $\Psi(MAT) = \Psi(MAT^\lambda)$.

Note that whether the classes MAT and MAT$^\lambda$ are equal is an open question in the theory of regulated rewriting.

The result $\Psi(MAT^\lambda) = \Psi(MAT)$ also means that the multiset language classes mARB and mMON coincide. mARB (mMON) is the class of multiset languages generated by arbitrary (monotone) multiset grammars (see [KMVP01] for details on multiset grammars). This result is due to the fact that $mARB = \Psi(MAT^\lambda)$ and $mMON = \Psi(MAT)$.

For more information on Hack’s conjecture, see [Gra79].

2 Petri net languages

A language of the form $L_\varphi$ in the next lemma will be needed in one of the later proofs.

**Lemma 1.** For any monoid-homomorphism $\varphi : \Sigma^* \to \mathbb{Z}$, the language $L_\varphi = \{w \in \Sigma^* \mid \varphi(w) \geq 0\}$ is in $\mathcal{L}_0$.

**Proof.** We construct a $\lambda$-free Petri net $N = (\Sigma, P, T, \partial_0, \partial_1, \sigma, \mu_0, F)$. Let $M := \max\{|\varphi(a)| \mid a \in \Sigma\}$, $\Sigma_- := \{a \in \Sigma \mid \varphi(a) < 0\}$, and $\Sigma_+ := \{a \in \Sigma \mid \varphi(a) \geq 0\}$. The set of transitions is

$$T := \{t_{a,r,s}^+ \mid a \in \Sigma_+, 0 \leq r \leq M, 0 \leq s \leq \varphi(a)\} \cup \{t_{a,r}^- \mid a \in \Sigma_-, 0 \leq r \leq M\}$$

and the set of places is $P = \{p_+, p_-\}$. For any $a \in \Sigma_+$, $0 \leq r \leq M$, $0 \leq s \leq \varphi(a)$, $\sigma(t_{a,r,s}^+) := a$ and for any $a \in \Sigma_-$, $0 \leq r \leq M$, $\sigma(t_{a,r}^-) := a$.

The initial marking is $\mu_0 := 0$ and the final markings are

$$F := \{r \cdot (p_+ + p_-) \mid 0 \leq r \leq M\}.$$ 

The net will work as follows. For $\mu \in P^\mathbb{N}$, let $\psi(\mu) := \mu(p_+) - \mu(p_-)$. If $\varphi(a) \geq 0$, the firing of a $a$-labeled transition $t_{a,r,s}^+$ increases the image of the marking under $\psi$ by a value between 0 and $\varphi(a)$. Furthermore, it subtracts $r$ from both $p_+$ and $p_-$. The latter does not change the image of the marking under $\psi$ but makes sure that the markings can be kept small.
If \( \varphi(a) < 0 \), then the transitions \( t_{a,r}^- \) add \( \varphi(a) \) to the image of the marking under \( \psi \). Besides, they subtract a certain value from both places. The pre- and post-multisets are as follows. For \( a \in \Sigma, 0 \leq r \leq M \), and, in case \( a \in \Sigma^+, 0 \leq s \leq \varphi(a) \), let

\[
\partial_0(t_{a,r,s}^+) := r \cdot (p_- + p_+), \quad \partial_1(t_{a,r,s}^+) := s \cdot p_+, \quad \text{for } a \in \Sigma^+,
\]

\[
\partial_0(t_{a,r}^-) := r \cdot (p_- + p_+), \quad \partial_1(t_{a,r}^-) := (-\varphi(a)) \cdot p_-, \quad \text{for } a \in \Sigma^-.
\]

Let \( \mu \in P^\oplus \) be reachable by a sequence labeled with \( w \in \Sigma^* \). By induction on the length of \( w \in \Sigma^* \), one can see that \( \psi(\mu) \leq \varphi(w) \). The fact that \( \psi(\mu) = 0 \) for every \( \mu \in F \) now shows that \( L(N) \subseteq L_\varphi \).

On the other hand, the following fact is also clear by induction on the length of \( w \). For every \( w \in \Sigma^* \), there is a firing sequence \( s, \sigma(s) = w \), that leads to a marking \( \mu \) such that \( \mu(p_+) \leq M \) and

- if \( \varphi(w) \geq 0 \), then \( \psi(\mu) = 0 \),
- if \( \varphi(w) < 0 \), then \( \psi(\mu) = \varphi(w) \).

Therefore, we have \( L_\varphi \subseteq L(N) \).

\[
3 \quad \text{Hack’s Conjecture}
\]

The equality \( \Psi(\text{MAT}) = \Psi(\text{MAT}^\lambda) \) can already be deduced from a slightly weaker version of Hack’s Conjecture, which will be stated here.

**Conjecture 2** (Hack’s Conjecture). For every Petri net \( N \), there is a constant \( K \in \mathbb{N} \) such that for any marking \( \mu \), the empty marking is reachable from \( \mu \) iff it can be reached by a firing sequence of length less than \( K \cdot ||\mu|| \).

The difference between this version and Hack’s version is that in the latter, the computability of the constant \( K_y \) from the size \( y \) is also stated. Furthermore, Conjecture 2 only requires the constant \( K \) to depend on \( N \), whereas Hack’s original conjecture states that the constant only depends on the size of \( N \). This, however, is not a weaker requirement, since, up to initial and final markings, there are only finitely many Petri nets of a certain size. Therefore, if there is such a constant for every Petri net, then there is a constant for every given size.

We will need the following result from the article [Zet09]. It states that applying linear erasing homomorphisms to languages generated by \( \lambda \)-free Petri nets yields languages that can be generated by non-erasing matrix grammars.
Theorem 3 ([Zet09]). $\mathcal{H}^{\text{lin}}(\mathcal{L}_0) \subseteq \text{MAT}$.

The next lemma states that arbitrary matrix languages and Petri net languages have the same Parikh image. For a proof, see [HJ94].

Lemma 4 ([HJ94]). $\Psi(\text{MAT}^\lambda) = \Psi(\mathcal{L}_0^\lambda)$.

The following is the key lemma in our proof, since it describes the consequences of Hack’s Conjecture in terms of Petri net languages.

Lemma 5. Suppose Hack’s Conjecture holds. Let $L \subseteq \Sigma^*$ be in $\mathcal{L}_0$ and $x \in \Sigma$ be a symbol. Then there is a $k \in \mathbb{N}$ such that for any word $w \in L \setminus \{x\}^*$, there is a $v \in L$ with $|v| \leq k \cdot |\delta_x(v)|$.

Proof. Let $N = (\Sigma, P, T, \partial_0, \partial_1, \sigma, \mu_0, F)$ be a $\lambda$-free Petri net such that $L(N) = L$ and let $K$ be the constant from Hack’s Conjecture. From $N$, we construct a Petri net $N' = (\Sigma', P', T', \partial'_0, \partial'_1, \sigma', \mu'_0, F')$, to which we will apply Hack’s Conjecture. Let $\Sigma' := \Sigma \setminus \{x\}$ and let $p_a$ for every $a \in \Sigma'$ and $r$ be new places. Furthermore, let $t_\mu$ be a new transition for every $\mu \in F$, and let $\mu'_0 := \mu_0 + r$. The new set of places is then $P' := P \cup \{p_a \mid a \in \Sigma' \} \cup \{r\}$ and the new set of transitions is $T' := T \cup \{t_\mu \mid \mu \in F\}$. For any $t \in T$ and any $\mu \in F$, let

$$
\delta'_0(t) := \begin{cases} r + \partial_0(t) + p_{\sigma(t)} & \text{if } \sigma(t) \neq x, \\
r + \partial_0(t) & \text{otherwise},
\end{cases} \quad \delta'_1(t) := r + \partial_1(t),
$$

$$
\delta'_0(t_\mu) := r + \mu, \quad \delta'_1(t_\mu) := 0, \\
\sigma'(t) := \sigma(t), \quad \sigma'(t_\mu) := \lambda.
$$

The embedding morphism $\iota : \Sigma^{\partial_0} \to P^{\partial_0}$ is defined by $\iota(a) := p_a$ for $a \in \Sigma'$. Since $w \notin \{x\}^*$, we have $|\delta_x(w)| \geq 1$. Since $w \in L$, there is a firing sequence $s$ in $N'$ that leads from $\nu_w := \mu'_0 + \iota(\Psi(\delta_x(w)))$ to $0$. It follows from the hypothesis that there is also a firing sequence $s'$ leading from $\nu_w$ to $0$ such that $|s'| < K \cdot |\nu_w|$. With $v := \sigma'(s')$, we have $\Psi(\delta_x(v)) = \Psi(\delta_x(w))$ and thus

$$
|v| \leq |s'| < K \cdot |\nu_w| = K \cdot (||\mu'_0|| + |\delta_x(w)|) = K \cdot (||\mu'_0|| + |\delta_x(v)|) \\
\leq K \cdot (||\mu'_0|| \cdot |\delta_x(v)| + |\delta_x(v)|) = K(||\mu'_0|| + 1) \cdot |\delta_x(v)|.
$$

Therefore, $k := K(||\mu'_0|| + 1) = K(||\mu_0|| + 2)$ meets our requirements.

We will now use the last lemma to infer an inclusion of multiset language classes.
Lemma 6. If Hack’s Conjecture holds, then $\Psi(L_0^\lambda) \subseteq \Psi(H_{\text{lin}}(L_0))$.

Proof. Let $L \subseteq \Sigma^*$ be in $L_0^\lambda$ and let $x \in \Sigma$ be a symbol that does not occur in $L$. Without loss of generality, we can assume that $\lambda \notin L$. Then write $L = \delta_x(M)$ for some $M \subseteq \Sigma^*$ in $L_0$. Note that $M \cap \{x\}^* = \emptyset$.

For $M$, Lemma 5 yields a $k$ with the property stated there. Let $R_{\Sigma,k}(x)$ be defined by

$$R_{\Sigma,k}(x) := \{ w \in \Sigma^* \mid |w| \leq k \cdot |\delta_x(w)| \},$$

which is in $L_0$ according to Lemma 1. Since $L_0$ is closed under intersection, the language $M' = M \cap R_{\Sigma,k}(x)$ is still in $L_0$. The property from Lemma 5 implies $\Psi(\delta_x(M)) = \Psi(\delta_x(M'))$. The homomorphism $\delta_x$ is linear erasing on $M' \subseteq R_{\Sigma,k}(x)$. Therefore, $\Psi(L) = \Psi(\delta_x(M)) = \Psi(\delta_x(M'))$ is in $\Psi(H_{\text{lin}}(L_0))$.

We are now ready to prove the main result.

Theorem 7. If Hack’s Conjecture holds, then $\Psi(\text{MAT}) = \Psi(\text{MAT}^\lambda)$.

Proof. $\Psi(\text{MAT}) \subseteq \Psi(\text{MAT}^\lambda)$ follows directly from the definition. Lemma 1, Lemma 6, and Theorem 3 imply

$$\Psi(\text{MAT}^\lambda) = \Psi(L_0^\lambda) \subseteq \Psi(H_{\text{lin}}(L_0)) \subseteq \Psi(\text{MAT}).$$

It is a well-known fact that $\Psi(\text{MAT}^\lambda) = \text{mARB}$ and $\Psi(\text{MAT}) = \text{mMON}$ (see [KMVP01, Theorem 1]). This yields the following corollary.

Corollary 8. If Hack’s Conjecture holds, then $\text{mARB} = \text{mMON}$.

References


