Chains in Collatz's Tree

Ştefan ANDREI [*], Manfred KUDLEK[†], Radu Ştefan NICULESCU[‡]

Abstract

The paper refers to the **Collatz's conjecture**. In the first part, we present some equivalent forms of this conjecture and a slight generalization of a former result from [AnM98]. Then, we present the notion of "chain subtrees" in Collatz's tree followed by a characterization theorem and some subclass of numbers which are labels for some chain subtrees. Next, we define the notion of "fixed points" and using this, we give another conjecture similar to Collatz's conjecture. Some new infinite sets of numbers for which the Collatz's conjecture holds are given. Finally, we present some interesting results related to the number of "even" and "odd" branches in the Collatz's tree.

Keywords: Collatz's Tree, chain subtrees, fixed points

Mathematics Subject Classification: 68Q25, 11-04, 11A07, 11A51

1 Introduction

The exact origin of the **Collatz's conjecture**, also called *Syracuse conjecture*, $3x + 1$ *problem*, *Kakutani's problem*, *Hasse algorithm*, and *Ulam's problem* is not clearly known. It circulated orally among the mathematical community for many years. This problem is credited to Lothar Collatz (University of Hamburg, [Col76, Col80]). In his student days in the 1930's, stimulated by the lectures of Edmund Landau, Oskar Perron, and Issai Schur, he became interested in number-theoretic functions. A lot of researchers have studied this conjecture ([Con72, Guy83, Lag85],...). Some prizes have been offered by researchers for its solution: \$50 by H. S. M. Coxeter in 1970 (and \$100 for a counterexample), then \$500 by Pál Erdős, and £1000 by B. Thwaites ([WTP82]).

*Faculty of Informatics, "Al.I.Cuza" University, Str. Berthelot, Nr. 16, 6600, Iaşi, România. E-mail: stefan@infoiasi.ro. This work was supported by The World Bank/Joint Japan Graduate Scholarship Program.

†Fachbereich Informatik, Universität Hamburg, Vogt-Kölln-Straße 30, D-22527 Hamburg, Germany. E-mail: kudlek@informatik.uni-hamburg.de

‡Faculty of Mathematics, Dept. of Computer Science, Bucharest University, Str. Academiei, Nr. 14, Sector 1, 70109, Bucuresti, România. E-mail: nick@colegiu.cs.unibuc.ro. This work has been done during a scholarship granted by Romanian Government.

Let $\mathbf{f}_{\mathcal{C}} : \mathbf{N}^* \to \mathbf{N}^*$ be the ("Collatz") function

$$\mathbf{f}_{\mathcal{C}}(n) = \begin{cases} \frac{n}{2} & \textit{if } n \equiv 0 \pmod 2 \\ 3n+1 & \textit{otherwise} \end{cases}$$

Show that for any $n \in \mathbf{N}^*$, there exists a $k \in \mathbf{N}^*$ such that $\mathbf{f}_{\mathcal{C}}{}^{(k)}(n) = 1$, $(\mathbf{f}_{\mathcal{C}}{}^{(1)} = \mathbf{f}_{\mathcal{C}},\ \mathbf{f}_{\mathcal{C}}{}^{(k+1)} = \mathbf{f}_{\mathcal{C}} \circ \mathbf{f}_{\mathcal{C}}{}^{(k)})$.

The conjecture may be rephrased as follows:
The following program halts for any given integer m:

```
n := m;
while n > 1 do
  if (n is even) then
    n := n/2
  else
    n := 3n + 1
  endif
endwhile;
```

■

This represents a useful example when we speak about terminating / non-terminating algorithms or about non-total recursive functions. The above algorithm obviously terminates for $m \leq 1$ (the body of the "while" loop is never executed) and presents no scientifical interest. More than that, if $m > 1$, the only way we can have a finite execution is to "reach" the value 1.

Notation 1.1 *We shall use the following notations:*

- $m \equiv n \pmod p$*, where* $m,\ n \in \mathbf{Z}$ *and* $p \in \mathbf{Z} - \{0\}$ *iff* $p \mid (m-n)$;
- $m \equiv \{n_1, ..., n_r\} \pmod p$*, where* $m,\ n_1, ...,\ n_r \in \mathbf{Z}$, $p \in \mathbf{Z} - \{0\}$ *and* $p \geq 2$ *means that* $m \equiv n_1 \pmod p$ *or ... or* $m \equiv n_r \pmod p$.

Theorem 1.1 *(a slight generalization of Theorem 4.2, [AnM98])*

(i) $\mathbf{f}_{\mathcal{C}}{}^{(3^m \cdot n + 2p)}\left(\frac{2^p \cdot \left(2^{3^m \cdot n}+1\right)}{3^p} - 1\right) = 1$, $\forall\ m \in \mathbf{N}$, $\forall\ n \equiv \{1,5\} \pmod 6$, $\forall\ p \in \{0, ..., m+1\}$*;*

(ii) $\mathbf{f}_{\mathcal{C}}{}^{(k+m)}(2^k \cdot n) = \mathbf{f}_{\mathcal{C}}{}^{(m)}(n),\ \forall\ m, n, k \in \mathbf{N}$;

(iii) $\mathbf{f}_{\mathcal{C}}{}^{(3^m \cdot n + 2p + k)}\left(2^k \cdot \left[\frac{2^p \cdot \left(2^{3^m \cdot n}+1\right)}{3^p} - 1\right]\right) = 1$, $\forall\ m \in \mathbf{N}$, $\forall\ n \equiv \{1,5\} \pmod 6$, $\forall\ p \in \{0, ..., m+1\}$, $\forall\ k \in \mathbf{N}$.

Proof

(i) Similarly to the proof of Theorem 4.2 from [AnM98]. According to Theorem 3.1 ([AnM98]) we deduce that

$$(*) \quad \mathbf{f}_{\mathcal{C}}{}^{(k+2p)}(2^p \cdot r - 1) = \mathbf{f}_{\mathcal{C}}{}^{(k)}(3^p \cdot r - 1), \ \forall\, p \in \mathbf{N},\ r \in \mathbf{N}_+$$

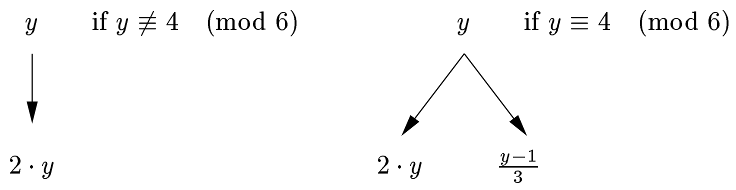
Solving the equation $3^p \cdot r - 1 = 2^k$, we obtain $r = \frac{2^{3^m \cdot n}+1}{3^p}$ (for $p = m+1$ follows the result from Theorem 4.2 [AnM98]). By replacing r in $(*)$ and due to the fact that $\mathbf{f}_{\mathcal{C}}{}^{(k)}(2^k) = 1$, we immediately obtain our identity.

(ii) Obviously, by induction on k. (iii) is obtained from (i) and (ii).

■

2 Binary Trees with Chains

We can associate to any $y \in \mathbf{N}$, $k \in \mathbf{N}$ a finite binary tree with root y, having k levels, denoted $A_k(y) = (V, E)$. (we can suppose $y \neq 0$ and $k \neq 0$). Each node $v \in V$ is labelled with a natural number. Each node may have one or two descendants (depending on the label !)



(a) (b)

The entire infinite tree for the root y will be $A(y) = (V, E)$, $V \subseteq \mathbf{N}$ (called **Collatz's tree of the root** y). Now, because of $\mathbf{f}_{\mathcal{C}}(1) = 4$, $\mathbf{f}_{\mathcal{C}}{}^{(2)}(1) = 2$, $\mathbf{f}_{\mathcal{C}}{}^{(3)}(1) = 1$ there is a "loop" in the Collatz's function causing repetition of 1 as a root of a subtree infinitely often. To avoid this fact, the Collatz function can be represented by another graph related to the infinite tree $A(8)$ attached to it the loop $8 \leftarrow 4 \leftarrow 2 \leftarrow 1 \leftarrow 4$. This is the only loop in that graph since all nodes in $A(8)$ have different labels, and they are not $1, 2, 4$.

We can reformulate Collatz's conjecture as:

$$V(A(8)) = \mathbf{N} - \{0, 1, 2, 4\}.$$

Lemma 2.1 *In $A(8)$ the labels of different vertices are distinct, and different from $1, 2, 4$.*

Proof By induction on the number of levels.
Basis: $A_1(8)$ has different labels.
Inductive Step: We know that the labels of different vertices from $A_k(y)$ are distinct. We have to prove that the labels from the next level are distinct from

the former ones and pairwise distinct, too. Let v be an arbitrary vertex from level $k+1$, labelled by y.

I: We suppose that there exists $w \in A_{k+1}(y)$, $w \neq v$ for which $label(w) = y$, too. Of course, $father(v) \in A_k(y)$, $father(w) \in A_k(y)$. Because of the fact that $2 * label(father(v)) \neq (label(father(w)) - 1)/3$, the case $father(v) = father(w)$ cannot occur. Supposing that $father(v) \neq father(w)$, we distinguish the cases:

a) y even. Then $label(father(v)) = \frac{label(y)}{2}$ and $label(father(w)) = \frac{label(y)}{2}$. Thus $label(father(v)) = label(father(w))$, thus a contradiction to the inductive hypothesis;

b) y odd. Then

$$label(father(v)) = 3 \cdot label(y) + 1 \text{ and } label(father(w)) = 3 \cdot label(y) + 1.$$

So, again $label(father(v)) = label(father(w))$, which is a contradiction to the inductive hypothesis.

II: There exists a node on level $m \leq k$ with label y. If $m > 0$ this contradicts the induction hypothesis. If $m = 0$ this means $y = 8$ and therefore a contradiction: $4 \in A(8)$. ■

As a consequence of Lemma 2.1, for generating the Collatz's tree, we don't have to store in memory the entire Collatz's tree. The last level suffices.

Since the **Collatz Conjecture** is still an open problem, it is possible that there exist other infinite connected Collatz graphs with a finite loop, and/or bi-infinite Collatz trees. In both cases, all nodes of such connected structures have different labels.

Lemma 2.2 *Let A be an infinite connected Collatz graph with a loop, constructed according to* **Collatz's** *tree. Then all nodes have different labels.*

Proof Let L be the finite loop. Trivially, all nodes of L have different labels. It is also evident from the Collatz function that there cannot exist edges entering the loop L.

Assume that there exists another finite loop L' in A. Since A is connected and any loop cannot have edges entering the only possibility for connection is that there exists a node s which is reached by paths both from L and L'. But this is a contradiction since no node can have more than one incoming edge.

To show that all nodes outside L have different labels, and also different from all labels from L, we use induction on the distance $d(u)$ for a node u from the loop L.

Basis: $d(u) = 1$ is obvious. Since the two children of a node $s \in L$ have different labels, and $label(u) = label(s)$ for some $s \in L$ would result in two incoming edges for s.

Inductive Step: Assume the induction hypothesis holds for all $t \in L$ with $d(u) \leq n$.

Let u be a node with $d(u) = n+1$ with $label(u) = label(v)$ for some other $v \in L$ with $d(v) \leq n+1$.

If $d(v) = n+1$ then we obtain $label(father(u)) = label(father(v))$ by the property of the Collatz function, and since $father(u) \neq father(v)$ this contradicts the induction hypothesis.

If $d(v) \leq n$ then $father(u) \neq father(v)$, but $label(u) = label(v)$, yielding again a contradiction to the induction hypothesis. ■

Lemma 2.3 *Let B be a bi-infinite connected* **Collatz** *tree (without loop). Then the labels of all nodes are distinct.*

Proof Assume the contrary.

a) If there exist two different nodes $u, v \in B$ for which $label(u) = label(v)$ on the same (infinite) path then that would be a loop, contradicting the assumption of loop-freeness.

b) If there exist two different nodes not on different path for which we have $label(u) = label(v)$ then there exists a common ancestor $t \in B$. Because of the **Collatz** function we get $label(father(u)) = label(father(v))$, and this can be continued.

This procedure either gives different s, t on the same path, thus yielding a loop. A contradiction to the assumption. Or it gives s, s' for which we have $father(s) = father(s') = t$ which is impossible since $label(s) \neq label(s')$. ■

Definition 2.1 *We say that $C = (V, E)$ is an* **(infinite) chain subtree of a tree** *T if C is a maximal subtree such that every node has (exactly) at most one direct descendant. We say that the node v is the root of C if $father(v)$ in T does not belong to $V(C)$.*

In fact, a chain subtree is formed only by using the situation (a) from the construction of Collatz's tree. If we know the label of the root of the chain subtree, it is obvious that we can deduce the labels of all the descendants (the labels are multiplied by 2).

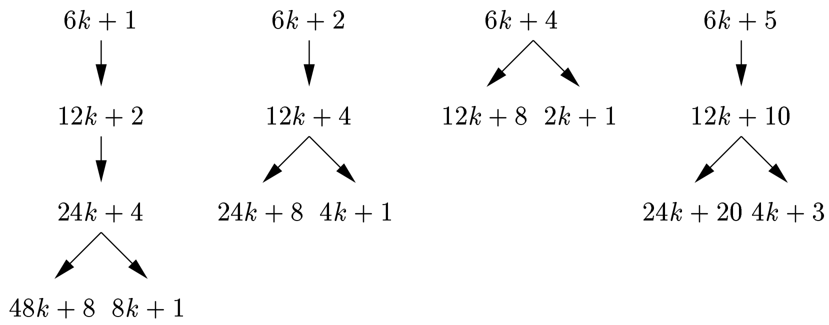
Theorem 2.1 *(characterization of chains in Collatz's tree)*
Let $A(y)$ be the Collatz's tree. Then:

(i) there exists a chain C in $A(y)$ with $v \in V(C)$ iff $3 \mid label(v)$;

(ii) much more, v is the root of C iff $3 \nmid label(father(v))$ if $father(v)$ exists in $A(y)$.

Proof

(i) ($\Longrightarrow$) $v \in V(C)$. If $3 \mid label(v)$ then we obtain the conclusion. Let us suppose, by contrary, that $3 \nmid label(v)$. So, the label of v is of the form $\{1, 2, 4, 5\} \pmod 6$. The following picture shows that each of these cases leads to a non-chain (i.e. there exists a descendant of v which has two sons).



($\Longleftarrow$) $3 \mid label(v)$. Let $label(v) = 2^k \cdot s$ where $3 \mid s$, $(2, s) = 1$, $k,\ s \in \mathbf{N}$. Let $k_0 \in \mathbf{N}$ be the minimum $t \in \mathbf{N},\ t \leq k$ such that there exists $u \in V(A(y))$ with $label(u) = 2^t \cdot s$.

Let u_0 be the corresponding u for k_0. We state that u_0 and its descendants form a chain C. We know that $3 \mid label(v)$, so $3 \mid u_0$. Therefore u_0 has exactly one direct descendent u_1 for which $label(u_1) = 2 \cdot label(u_0)$. Thus $3 \mid label(u_1)$, and u_1 has exactly one descendant u_2 etc. This means that C is a subtree of $A(y)$ in which every node has exactly one direct descendant. If C hadn't been maximal we could have added $father(u_0)$ to C. Because of the minimality of k_0, we have $label(father(u_0)) \neq 2^{k_0-1} \cdot s$. Hence $k_0 = 0$, so $label(u_0) = s$. Therefore $label(father(u_0)) = 3s + 1$. Because $(s, 2) = 1$, it follows that $3s + 1 \equiv 4 \pmod 6$, so $father(u_0)$ has two direct descendants. Contradiction ! This implies that C is maximal, i.e. a chain.

(ii) ($\Longrightarrow$) v is the root of C. If $3 \mid label(father(v))$ then C is not maximal (because we can add $father(v)$ to C). This is a contradiction to Definition 2.1.

($\Longleftarrow$) $3 \not| \, label(father(v))$. If v is not the root of C then there exists $w \in C$ such that v is the son of w. Because w belongs to $V(A(y))$, it follows that w is the father of v in $A(y)$. But from $w \in C$ we have $3 \mid label(w)$. This is a contradiction to $3 \not| \, label(father(v))$.

■

The following figure shows after how many levels (for a given node), there exists a node labelled by a multiple of 3.

According to the figure below and Theorem 2.1, for a vertex labelled by y, after at most k levels, we obtain a chain tree in $A(y)$.

y	3k	9k+5	9k+7	9k+8	9k+4	9k+2	9k+1
k	0	2	3	4	5	6	7

The roots of chain trees are pointed out in the below figure by putting their label into a rectangle.

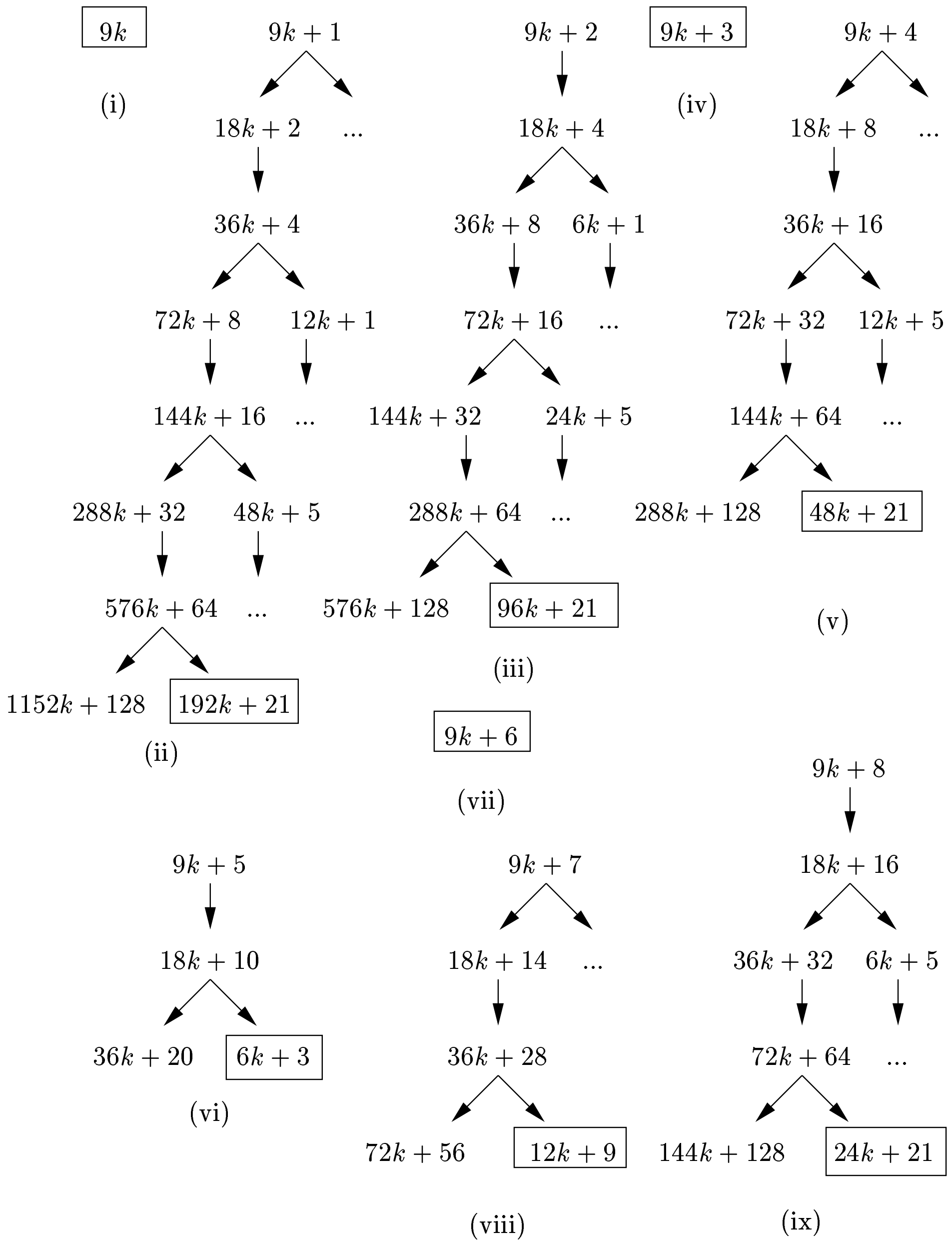


Figure 2. Nine situations for obtaining chain subtrees

Lemma 2.4 *The following facts hold:*

a) $2^{2k} \equiv 1 \pmod 3$ *and* $2^{2k+1} \equiv 2 \pmod 3$, $\forall\ k \in \mathbf{N}$;

b) $\frac{2^{3^m}+1}{3^{m+1}} \equiv 1 \pmod 3,\ \forall\, m \in \mathbf{N};$

c) $\frac{2^{3^m \cdot n}+1}{3^{m+1}} \equiv n \pmod 3,\ \forall\, m \in \mathbf{N},\ \forall\, n \equiv \{1,5\} \pmod 6.$

Proof
a) Obviously, $4 = 2^2 \equiv 1 \pmod 3$. Therefore $\left(2^2\right)^k = 2^{2k} \equiv 1 \pmod 3$ and $2^{2k+1} \equiv 2 \pmod 3,\ \forall\, k \in \mathbf{N};$
b) $2^{3^m} = (3-1)^{3^m} = \sum\limits_{i=0}^{3^m} \begin{pmatrix} 3^m \\ i \end{pmatrix} \cdot 3^i \cdot (-1)^{3^m - i} = -1 + 3^{m+1} + 3^{m+2} \cdot M$, where M is obtained from the terms corresponding to $i = 2, ..., 3^m$. Therefore, $\frac{2^{3^m}+1}{3^{m+1}} \equiv 1 \pmod 3,\ \forall\, m \in \mathbf{N};$
c) $2^{3^m \cdot n} = (3-1)^{3^m \cdot n} = \sum\limits_{i=0}^{3^m \cdot n} \begin{pmatrix} 3^m \cdot n \\ i \end{pmatrix} \cdot 3^i \cdot (-1)^{3^m \cdot n - i}$. This equals to $-1 + 3^{m+1} \cdot n + 3^{m+2} \cdot n \cdot M$, where M is obtained from the terms corresponding to $i = 2, ..., 3^m$. Therefore, $\frac{2^{3^m \cdot n}+1}{3^{m+1}} \equiv n \pmod 3,\ \forall\, m \in \mathbf{N}, \forall\, n \equiv \{1,5\} \pmod 6$. ■

Theorem 2.2 *The following identity holds:*

$$\mathbf{f}_C{}^{(3^m \cdot n + 2m + k + 2)} \left(2^k \cdot \left[\frac{2^{m+1} \cdot (2^{3^m \cdot n} + 1)}{3^{m+1}} - 1\right]\right) = 1,$$

$\forall\, m \in \mathbf{N},\ n \equiv \{1,5\} \pmod 6,\ \forall\, k \in \mathbf{N}$. *Much more, if* $m \equiv 1 \pmod 2$ *and* $n \equiv 1 \pmod 6$*, or* $m \equiv 0 \pmod 2$ *and* $n \equiv 5 \pmod 6$*, then these numbers (for which Collatz's conjecture holds) belong to a chain subtree in Collatz's tree. The label of the root may be obtained by taking* $k = 0$.

Proof The first part can be obviously obtained by taking $p = m+1$ in Theorem 1.1.

For the second part, we shall show that $\frac{2^{m+1} \cdot (2^{3^m \cdot n}+1)}{3^{m+1}} - 1 \equiv 0 \pmod 3$. Then, applying Theorem 2.1, we obtain that these numbers belong to a chain.

Because $2 \equiv -1 \pmod 3$, we get $2^{m+1} \equiv (-1)^{m+1} \pmod 3$. According to Lemma 2.4, c), we obtain $\frac{2^{m+1} \cdot (2^{3^m \cdot n}+1)}{3^{m+1}} \equiv (-1)^{m+1} \cdot n \pmod 3$.

Now, if $m \equiv 1 \pmod 2$ and $n \equiv 1 \pmod 6$ then $(-1)^{m+1} \cdot n \equiv 1 \pmod 3$. The same relation may be obtained in the other case ($m \equiv 0 \pmod 2$ and $n \equiv 5 \pmod 6$), too. ■

Theorem 2.3 *Let* $m \in \mathbf{N}$, $n \equiv \{1,5\} \pmod 6$, $p \in \{1, ..., m-1\}$, $s \in \mathbf{N}$ *arbitrary natural numbers. Then (depending on k) the following identities hold:*

(i) If $k \equiv 0 \pmod 6$ *then*

$$\mathbf{f}_C{}^{(3^m \cdot n + 2p + k + s + 4)} \left(2^s \cdot \left[\frac{2^{p+k+3} \cdot \left(2^{3^m \cdot n} + 1\right) - \left(2^{k+3} + 1\right) \cdot 3^p}{3^{p+1}}\right]\right) = 1;$$

(ii) If $k \equiv 1 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}^{(3^m \cdot n+2p+k+s+3)}\left(2^s \cdot \left[\frac{2^{p+k+2}\cdot\left(2^{3^m\cdot n}+1\right)-\left(2^{k+2}+1\right)\cdot 3^p}{3^{p+1}}\right]\right)=1;$$

(iii) If $k \equiv 2 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}^{(3^m \cdot n+2p+k+s+2)}\left(2^s \cdot \left[\frac{2^{p+k+1}\cdot\left(2^{3^m\cdot n}+1\right)-\left(2^{k+1}+1\right)\cdot 3^p}{3^{p+1}}\right]\right)=1;$$

(iv) If $k \equiv 3 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}^{(3^m \cdot n+2p+k+s+7)}\left(2^s \cdot \left[\frac{2^{p+k+6}\cdot\left(2^{3^m\cdot n}+1\right)-\left(2^{k+6}+1\right)\cdot 3^p}{3^{p+1}}\right]\right)=1;$$

(v) If $k \equiv 4 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}^{(3^m \cdot n+2p+k+s+6)}\left(2^s \cdot \left[\frac{2^{p+k+5}\cdot\left(2^{3^m\cdot n}+1\right)-\left(2^{k+5}+1\right)\cdot 3^p}{3^{p+1}}\right]\right)=1;$$

(vi) If $k \equiv 5 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}^{(3^m \cdot n+2p+k+s+5)}\left(2^s \cdot \left[\frac{2^{p+k+4}\cdot\left(2^{3^m\cdot n}+1\right)-\left(2^{k+4}+1\right)\cdot 3^p}{3^{p+1}}\right]\right)=1.$$

Much more, these numbers belong to a chain subtree in Collatz's tree. The label of the root may be obtained by taking $s = 0$.

Proof We know from Theorem 1.1 that:

$$\mathbf{f}_{\mathcal{C}}^{(3^m \cdot n+2p+k)}\left(2^k \cdot \left[\frac{2^p\cdot\left(2^{3^m\cdot n}+1\right)}{3^p}-1\right]\right)=1,$$

$\forall\, m \in \mathbf{N}$, $\forall\, n \equiv \{1,5\} \pmod 6$, $\forall\, k \in \mathbf{N}$. We consider $p \in \{0,\ldots,m-1\}$.

From Lemma 2.4, c), we know that $\frac{2^{3^m\cdot n}+1}{3^{m+1}}$ is a natural number. So, $\frac{2^{3^m\cdot n}+1}{3^p} \equiv 0 \pmod 9$, $\forall\, p < m$ (and also $\frac{2^p\cdot(2^{3^m\cdot n}+1)}{3^p} \equiv 0 \pmod 9$, $\forall\, p<m$). Therefore $\frac{2^p\cdot\left(2^{3^m\cdot n}+1\right)}{3^p}-1 \equiv 8 \pmod 9$. Now, we have:

$$2^k = \begin{cases} 1 \pmod 9 & \text{if } k \equiv 0 \pmod 6 \\ 2 \pmod 9 & \text{if } k \equiv 1 \pmod 6 \\ 4 \pmod 9 & \text{if } k \equiv 2 \pmod 6 \\ 8 \pmod 9 & \text{if } k \equiv 3 \pmod 6 \\ 7 \pmod 9 & \text{if } k \equiv 4 \pmod 6 \\ 5 \pmod 9 & \text{if } k \equiv 5 \pmod 6 \end{cases}$$

Next, we shall show how we obtain (i) from our theorem, the other cases being analogous. For $k \equiv 0 \pmod 6$ we obtain $2^k \cdot \left[\frac{2^p \cdot (2^{3^m \cdot n}+1)}{3^p} - 1\right] \equiv 8 \pmod 9$. But from the figure 2, (ix), we remark that $\mathbf{f}_{\mathcal{C}}{}^{(4)}(24r+21) = 9r+8$. By replacing $9r+8$ with $2^k \cdot \left[\frac{2^p \cdot (2^{3^m \cdot n}+1)}{3^p} - 1\right]$ we can obtain:

$$r = \frac{2^{p+k} \cdot \left(2^{3^m \cdot n} + 1\right) - 2^k \cdot 3^p - 2^3 \cdot 3^p}{3^{p+2}}.$$

Now, we obtain $24r + 21 = \frac{2^{p+k+3} \cdot (2^{3^m \cdot n}+1) - (2^{k+3}+1) \cdot 3^p}{3^{p+1}} \equiv 0 \pmod 3$. Therefore

$$\mathbf{f}_{\mathcal{C}}{}^{(3^m \cdot n+2p+k+4)} \left(\frac{2^{p+k+3} \cdot \left(2^{3^m \cdot n} + 1\right) - \left(2^{k+3} + 1\right) \cdot 3^p}{3^{p+1}} \right) =$$

$$= \mathbf{f}_{\mathcal{C}}{}^{(3^m \cdot n+2p+k)} \left(2^k \cdot \left[\frac{2^p \cdot \left(2^{3^m} + 1\right)}{3^p} - 1 \right] \right) = 1.$$

From this follows the conclusion by applying Theorems 1.1 (ii), and 2.1. ■

Theorem 2.4 *Let* $m \in \mathbf{N}$, $n \equiv \{1,5\} \pmod 6$, $s \in \mathbf{N}$ *arbitrary natural numbers. Then (depending on m, n and k) the following identities hold:*

a) If ($m \equiv \{0,2,4\} \pmod 9$ *and* $n \equiv 1 \pmod 6$) *or* ($m \equiv \{1,3,5\} \pmod 9$ *and* $n \equiv 5 \pmod 6$) *then*

(i) If $k \equiv 0 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}{}^{(3^m \cdot n+2m+k+s+6)} \left(2^s \cdot \left[\frac{2^{m+k+5} \cdot \left(2^{3^m \cdot n} + 1\right) - \left(2^{k+5} + 1\right) \cdot 3^m}{3^{m+1}} \right] \right) = 1;$$

(ii) If $k \equiv 1 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}{}^{(3^m \cdot n+2m+k+s+5)} \left(2^s \cdot \left[\frac{2^{m+k+4} \cdot \left(2^{3^m \cdot n} + 1\right) - \left(2^{k+4} + 1\right) \cdot 3^m}{3^{m+1}} \right] \right) = 1;$$

(iii) If $k \equiv 2 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}{}^{(3^m \cdot n+2m+k+s+4)} \left(2^s \cdot \left[\frac{2^{m+k+3} \cdot \left(2^{3^m \cdot n} + 1\right) - \left(2^{k+3} + 1\right) \cdot 3^m}{3^{m+1}} \right] \right) = 1;$$

(iv) If $k \equiv 3 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}{}^{(3^m \cdot n+2m+k+s+3)} \left(2^s \cdot \left[\frac{2^{m+k+2} \cdot \left(2^{3^m \cdot n} + 1\right) - \left(2^{k+2} + 1\right) \cdot 3^m}{3^{m+1}} \right] \right) = 1;$$

(v) If $k \equiv 4 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}{}^{(3^m\cdot n+2m+k+s+2)}\left(2^s\cdot\left[\frac{2^{m+k+1}\cdot\left(2^{3^m\cdot n}+1\right)-\left(2^{k+1}+1\right)\cdot 3^m}{3^{m+1}}\right]\right)=1;$$

(vi) If $k \equiv 5 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}{}^{(3^m\cdot n+2m+k+s+7)}\left(2^s\cdot\left[\frac{2^{m+k+6}\cdot\left(2^{3^m\cdot n}+1\right)-\left(2^{k+6}+1\right)\cdot 3^m}{3^{m+1}}\right]\right)=1.$$

b) If ($m \equiv \{0,2,4\} \pmod 9$ *and* $n \equiv 5 \pmod 6$*) or (*$m \equiv \{1,3,5\} \pmod 9$ *and* $n \equiv 1 \pmod 6$*) then*

(i) If $k \equiv 0 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}{}^{(3^m\cdot n+2m+k+s+2)}\left(2^s\cdot\left[\frac{2^{m+k+1}\cdot\left(2^{3^m\cdot n}+1\right)-\left(2^{k+1}+1\right)\cdot 3^m}{3^{m+1}}\right]\right)=1;$$

(ii) If $k \equiv 1 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}{}^{(3^m\cdot n+2m+k+s+7)}\left(2^s\cdot\left[\frac{2^{m+k+6}\cdot\left(2^{3^m\cdot n}+1\right)-\left(2^{k+6}+1\right)\cdot 3^m}{3^{m+1}}\right]\right)=1;$$

(iii) If $k \equiv 2 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}{}^{(3^m\cdot n+2m+k+s+6)}\left(2^s\cdot\left[\frac{2^{m+k+5}\cdot\left(2^{3^m\cdot n}+1\right)-\left(2^{k+5}+1\right)\cdot 3^m}{3^{m+1}}\right]\right)=1;$$

(iv) If $k \equiv 3 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}{}^{(3^m\cdot n+2m+k+s+5)}\left(2^s\cdot\left[\frac{2^{m+k+4}\cdot\left(2^{3^m\cdot n}+1\right)-\left(2^{k+4}+1\right)\cdot 3^m}{3^{m+1}}\right]\right)=1;$$

(v) If $k \equiv 4 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}{}^{(3^m\cdot n+2m+k+s+4)}\left(2^s\cdot\left[\frac{2^{m+k+3}\cdot\left(2^{3^m\cdot n}+1\right)-\left(2^{k+3}+1\right)\cdot 3^m}{3^{m+1}}\right]\right)=1;$$

(vi) If $k \equiv 5 \pmod 6$ *then*

$$\mathbf{f}_{\mathcal{C}}{}^{(3^m\cdot n+2m+k+s+3)}\left(2^s\cdot\left[\frac{2^{p+k+2}\cdot\left(2^{3^m\cdot n}+1\right)-\left(2^{k+2}+1\right)\cdot 3^m}{3^{m+1}}\right]\right)=1.$$

Much more, these numbers belong to a chain tree. The label of the root may be obtained taking $s = 0$.

Proof We know from Theorem 1.1 that:

$$\mathbf{f_\mathcal{C}}^{(3^m\cdot n+2m+k)}\left(2^k\cdot\left[\frac{2^m\cdot(2^{3^m\cdot n}+1)}{3^m}-1\right]\right)=1,$$

$\forall\, m\in\mathbf{N}$, $\forall\, n\equiv\{1,5\} \pmod 6$, $\forall\, k\in\mathbf{N}$. From Lemma 2.4, c), we know that $\frac{2^{3^m\cdot n}+1}{3^{m+1}}\equiv n \pmod 3$. So, $\frac{2^{3^m\cdot n}+1}{3^m}\equiv 3\cdot n \pmod 9$. Next, all the identities from the conclusion of this theorem can be obtained using a similar procedure as in the proof of Theorem 2.3. ∎

3 Some "fixed points"

In this section, we shall define some kind of "fixed points". Using this notion, we shall give a conjecture which is similar to the Collatz's Conjecture. Starting from numbers of the form $2^k\cdot s-t$, (t odd), our aim is to reduce them to $3^{k'}\cdot s-t'$, (t' odd) for which $t'\le t$ such that we could analyse solutions of $3^{k'}\cdot s-t'=2^p$ (such numbers can be reduced to 1).

Lemma 3.1 *Let k, s be arbitrary natural numbers ($k,\ s\ge 1$). Then the following idendities hold:*

(i) $\mathbf{f_\mathcal{C}}^{(2m)}(2^k\cdot s-1)=3^m\cdot 2^{k-m}\cdot s-1,\ \forall\, m\in\{0,...,k\}$;

(ii) $\mathbf{f_\mathcal{C}}^{(5m)}(2^k\cdot s-5)=3^{2m}\cdot 2^{k-3m}\cdot s-5,\ \forall\, m\in\{0,...,\lfloor\frac{k}{3}\rfloor\}$;

(iii) $\mathbf{f_\mathcal{C}}^{(18m)}(2^k\cdot s-17)=3^{7m}\cdot 2^{k-11m}\cdot s-17,\ \forall\, m\in\{0,...,\lfloor\frac{k}{11}\rfloor\}$.

Proof We proceed by induction on m.

(i) **Basis:** $m=1$. It is obvious that $\mathbf{f_\mathcal{C}}^{(2)}(2^k\cdot s-1)=3\cdot 2^{k-1}\cdot s-1$.

Inductive Step: We suppose that (i) is true for m and prove it for $m+1$ ($m+1\le k$). From the inductive hypothesis, we know that:

$$\mathbf{f_\mathcal{C}}^{(2m)}(2^k\cdot s-1)=3^m\cdot 2^{k-m}\cdot s-1.$$

But $\mathbf{f_\mathcal{C}}^{(2)}(3^m\cdot 2^{k-m}\cdot s-1)=\mathbf{f_\mathcal{C}}(3^{m+1}\cdot 2^{k-m}\cdot s-2)=3^{m+1}\cdot 2^{k-m-1}\cdot s-1$. Therefore $\mathbf{f_\mathcal{C}}^{(2m+2)}(2^k\cdot s-1)=3^{m+1}\cdot 2^{k-m-1}\cdot s-1$.

(ii) **Basis:** $m=1$. We can immediatelly remark that

$$\mathbf{f_\mathcal{C}}^{(5)}(2^k\cdot s-5)=\mathbf{f_\mathcal{C}}^{(4)}(3\cdot 2^k\cdot s-14)=\mathbf{f_\mathcal{C}}^{(3)}(3\cdot 2^{k-1}\cdot s-7)=$$

$$=\mathbf{f_\mathcal{C}}^{(2)}(3^2\cdot 2^{k-1}\cdot s-20)=\mathbf{f_\mathcal{C}}(3^2\cdot 2^{k-2}\cdot s-10)=3^2\cdot 2^{k-3}\cdot s-5.$$

Inductive Step: We suppose that (ii) is true for m and prove it for $m+1$ $(m+1 \leq \lfloor \frac{k}{3} \rfloor)$. From the inductive hypothesis, we know that:

$$\mathbf{f}_{\mathcal{C}}{}^{(5m)}(2^k \cdot s - 5) = 3^{2m} \cdot 2^{k-3m} \cdot s - 5.$$

But $\mathbf{f}_{\mathcal{C}}{}^{(5)}(3^{2m} \cdot 2^{k-3m} \cdot s - 5) = 3^{2m+2} \cdot 2^{k-3m-3} \cdot s - 5$. Therefore, we have $\mathbf{f}_{\mathcal{C}}{}^{(5m+5)}(2^k \cdot s - 5) = 3^{2m+2} \cdot 2^{k-3m-3} \cdot s - 5$.

(iii) **Basis:** $m = 1$. We can immediatelly remark that

$$\mathbf{f}_{\mathcal{C}}{}^{(18)}(2^k \cdot s - 17) = \mathbf{f}_{\mathcal{C}}{}^{(17)}(3 \cdot 2^k \cdot s - 50) = \mathbf{f}_{\mathcal{C}}{}^{(16)}(3 \cdot 2^{k-1} \cdot s - 25) =$$

$$= \mathbf{f}_{\mathcal{C}}{}^{(15)}(3^2 \cdot 2^{k-1} \cdot s - 74) = \mathbf{f}_{\mathcal{C}}{}^{(14)}(3^2 \cdot 2^{k-2} \cdot s - 37) = \mathbf{f}_{\mathcal{C}}{}^{(13)}(3^2 \cdot 2^{k-3} \cdot s - 110) =$$

$$= \mathbf{f}_{\mathcal{C}}{}^{(12)}(3^3 \cdot 2^{k-3} \cdot s - 55) = \mathbf{f}_{\mathcal{C}}{}^{(11)}(3^4 \cdot 2^{k-3} \cdot s - 164) = \mathbf{f}_{\mathcal{C}}{}^{(10)}(3^4 \cdot 2^{k-4} \cdot s - 82) =$$

$$= \mathbf{f}_{\mathcal{C}}{}^{(9)}(3^4 \cdot 2^{k-5} \cdot s - 41) = \mathbf{f}_{\mathcal{C}}{}^{(8)}(3^5 \cdot 2^{k-5} \cdot s - 122) = \mathbf{f}_{\mathcal{C}}{}^{(7)}(3^5 \cdot 2^{k-6} \cdot s - 61) =$$

$$= \mathbf{f}_{\mathcal{C}}{}^{(6)}(3^6 \cdot 2^{k-6} \cdot s - 182) = \mathbf{f}_{\mathcal{C}}{}^{(5)}(3^6 \cdot 2^{k-7} \cdot s - 91) = \mathbf{f}_{\mathcal{C}}{}^{(4)}(3^7 \cdot 2^{k-7} \cdot s - 272) =$$

$$= \mathbf{f}_{\mathcal{C}}{}^{(3)}(3^7 \cdot 2^{k-8} \cdot s - 136) = \mathbf{f}_{\mathcal{C}}{}^{(2)}(3^7 \cdot 2^{k-9} \cdot s - 68) = \mathbf{f}_{\mathcal{C}}(3^7 \cdot 2^{k-10} \cdot s - 34) =$$

$$= 3^7 \cdot 2^{k-11} \cdot s - 17$$

Inductive Step: We suppose that (iii) is true for m and prove it for $m+1$ $(m+1 \leq \lfloor \frac{k}{11} \rfloor)$. From the inductive hypothesis, we know that:

$$\mathbf{f}_{\mathcal{C}}{}^{(18m)}(2^k \cdot s - 17) = 3^{7m} \cdot 2^{k-11m} \cdot s - 17.$$

But $\mathbf{f}_{\mathcal{C}}{}^{(18)}(3^{7m} \cdot 2^{k-11m} \cdot s - 17) = 3^{7m+7} \cdot 2^{k-11m-11} \cdot s - 17$. Therefore $\mathbf{f}_{\mathcal{C}}{}^{(7m+7)}(2^k \cdot s - 17) = 3^{7m+7} \cdot 2^{k-11m-11} \cdot s - 17$.

■

Theorem 3.1 *The following identities hold (for all m and s natural numbers):*

(i) $\mathbf{f}_{\mathcal{C}}{}^{(2m)}(2^m \cdot s - 1) = 3^m \cdot s - 1$;

(ii) $\mathbf{f}_{\mathcal{C}}{}^{(5m)}(2^{3m} \cdot s - 5) = 3^{2m} \cdot s - 5$;

(iii) $\mathbf{f}_{\mathcal{C}}{}^{(18m)}(2^{11m} \cdot s - 17) = 3^{7m} \cdot s - 17$.

Proof We take in Lemma 3.1, $k = m$, $k = 3m$ and $k = 11m$ respectively. ■

Let us denote by

$$\mathcal{T} = \{t \mid \exists\, m, \exists\, k_1, \exists\, k_2,\ \exists\, t' \leq t \text{ such that } \mathbf{f}_{\mathcal{C}}{}^{(m)}(2^k \cdot s - t) = 3^{k_1} \cdot 2^{k_2} \cdot s - t'$$

$$\text{and } m \text{ is minimal with this property}\},$$

where k and s are arbitrary natural numbers. According to Theorem 3.1, we obtain $1,\ 5,\ 17 \in \mathcal{T}$ and much more $t' = t$. We may call these numbers $(1, 5, 17)$

fixed points. We checked on the computer for all $t \leq 10^8$ the membership to $\mathcal{T}$, and we obtained a positive answer. For the numbers less than 10^8 but which don't belong to $\{1, 5, 17\}$, we obtained $t' < t$. This means that these numbers are not fixed points. We conjecture that:

$$(C) \qquad \text{All natural numbers belong to } \mathcal{T}.$$

In the following, we shall prove that the equations of the form:

$$3^m \cdot s - 1 = 2^p,\ 3^{2m} \cdot s - 5 = 2^p,\ 3^{7m} \cdot s - 17 = 2^p$$

have always solutions. In addition, for every $m_0 \in \mathbf{N}$ we have an infinity of solutions (m, s, p) with $m = m_0$.

Because of this, the Collatz's Conjecture is similar to our conjecture (C), which can be reformulated in the following way:

Let $g : \mathbf{N}^* \to \mathbf{N}^*$ be the function

$$g(n) = \begin{cases} \frac{n}{2} & \textit{if } n \equiv 0 \pmod 2 \\ 3n - 1 & \textit{otherwise} \end{cases}$$

Show that, for any $n \in \mathbf{N}^*$, there exists an $k \in \mathbf{N}^*$ such that $g^{(k)}(n) \leq n$.

We shall prove some results that will help us further in our consideration. We shall use the following notations:

- $\widehat{x}_{3^k} = \{n \in \mathbf{N} \mid n \equiv x \pmod{3^k}\}$;
- $\mathbf{Z}_{3^k} = \{\widehat{x}_{3^k} \mid x \in \mathbf{N}\}$;
- $(\mathbf{Z}_{3^k}, \oplus, \odot)$ is a ring with $\oplus$ and $\odot$ defined below:
  - $\widehat{x}_{3^k} \oplus \widehat{y}_{3^k} = (\widehat{x + y})_{3^k}$;
  - $\widehat{x}_{3^k} \odot \widehat{y}_{3^k} = (\widehat{xy})_{3^k}$.
- $\mathbf{U}(\mathbf{Z}_{3^k})$ the group of invertible elements in $(\mathbf{Z}_{3^k}, \oplus, \odot)$, this means that $\mathbf{U}(\mathbf{Z}_{3^k}) = \{\widehat{x}_{3^k} \mid (x, 3^k) = 1,\ x \in \mathbf{N}\}$;
- $< x >$ the subgroup generated by x.

We shall use the following lemma.

Lemma 3.2 *Let α be the order of $\widehat{2}_{3^k}$ in $\mathbf{U}(\mathbf{Z}_{3^k})$, where $k \geq 1$. Then for every n such that $3^k \mid (2^n - 1)$ we have $\alpha \mid n$.*

Proof By definition, α is the least number β $(\beta \in \mathbf{N} - \{0\})$ with $\widehat{2}^{\beta}_{3^k} = \widehat{1}_{3_k}$ in $\mathbf{Z}_{3^k}$ (this is equivalent to $2^\beta \equiv 1 \pmod{3^k}$ (1)). If $n = \alpha t + r$ with $0 \leq r \leq \alpha - 1,\ t,\ r \in \mathbf{N}$, we have $2^n \equiv 2^{\alpha t} \cdot 2^r \equiv 2^r \pmod{3^k}$. For $3^k \mid (2^n - 1)$ we have $2^n \equiv 1 \pmod{3^k}$, so therefore $2^r \equiv 1 \pmod{3^k}$. If $r \neq 0$, this is a contradiction with (1) because $r < \alpha$. From this, we infer that $r = 0 \Longrightarrow n = \alpha t$. Thus $\alpha \mid n$ and the proof of the lemma is complete. ■

Theorem 3.2 $\mathbf{U}(\mathbf{Z}_{3^k}) =< \widehat{2}_{3^k} >$, *for all* $k \geq 1$.

Proof Let α be the order of $\widehat{2}_{3^k}$ in the group $\mathbf{U}(\mathbf{Z}_{3^k})$ ($\widehat{2}_{3^k} \in \mathbf{U}(\mathbf{Z}_{3^k})$ because 2 and 3 are mutually prime numbers). From Euler's Theorem it follows that $2^{\varphi(3^k)} \equiv 1 \pmod{3^k}$. This is equivalent to $2^{2\cdot 3^{k-1}} \equiv 1 \pmod{3^k}$. From the definition of α and Lemma 3.2, it follows that $\alpha \mid (2 \cdot 3^{k-1})$. From $2^\alpha \equiv 1 \pmod{3^k}$ we infer that $\alpha \equiv 0 \pmod{2}$. Therefore $\alpha = 2\cdot 3^t$, where $0 \leq t \leq k-1$.

Let us suppose that $t \leq k-2$. We have $2^{2\cdot 3^t} \equiv 1 \pmod{3^k}$. Therefore $\left(2^{2\cdot 3^t}\right)^{3^{(k-2)-t}} \equiv 1 \pmod{3^k}$. Thus $2^{2\cdot 3^{k-2}} \equiv 1 \pmod{3^k}$, so it follows $(2^{3^{k-2}}+1)(2^{3^{k-2}}-1) \equiv 0 \pmod{3^k}$. However $2^{3^{k-2}} - 1 \equiv 1 \pmod{3}$, that means $3^k \mid (2^{3^{k-2}}+1)$. This is an obvious contradiction with Lemma 2.4.

Hence, $t = k-1$. This implies $\alpha = 2 \cdot 3^{k-1} = card(\mathbf{U}(\mathbf{Z}_{3^k}))$, i.e. $\mathbf{U}(\mathbf{Z}_{3^k})$ is a cyclic group generated by $\widehat{2}_{3^k}$. ■

Theorem 3.3 *Let* u, v *be two given natural numbers with* $(3, v) = 1$. *Then for every* $m \in \mathbf{N}$ *the equation*

$$3^{um} \cdot s = 2^p + v \qquad (**)$$

satisfies:

a) *there exists a unique solution* (s_0, p_0) *of* $(**)$ *with* $0 \leq p_0 < 2 \cdot 3^{um-1}$;

b) *every solution* (s, p) *of the equation (**) with* $m > 0$ *is of the form*

$$\left(\frac{2^{p_0 + 2\cdot 3^{um-1}\cdot t} + v}{3^{um}}, p_0 + 2 \cdot 3^{um-1} \cdot t \right)$$

where $t \in \mathbf{N}$, *and viceversa, for every* $t \in \mathbf{N}$ *the above pair is a solution for* $(**)$.

Proof

a) We use Theorem 3.2 with $k = um$ and obtain $\mathbf{U}(\mathbf{Z}_{3^{um}}) =< \widehat{2}_{3^{um}} >$. Using $(3, v) = 1$, we obtain $-\widehat{v}_{3^{um}} \in \mathbf{U}(\mathbf{Z}_{3^{um}})$. Therefore there exists a unique p_0, $0 \leq p_0 < 2 \cdot 3^{um-1}$ ($2 \cdot 3^{um-1} = card(\mathbf{U}(\mathbf{Z}_{3^{um}}))$) such that $\widehat{2}_{3^{um}} = -\widehat{v}_{3^{um}}$ in $\mathbf{Z}_{3^{um}}$. This is equivalent to the fact that there exists a unique p_0, $p_0 < 2 \cdot 3^{um-1}$ such that $3^{um} \mid 2^p + v$, i.e. there exists a unique solution (s_0, p_0) for $(**)$ which satisfies $0 \leq p_0 < 2 \cdot 3^{um-1}$.

b) "$\Longrightarrow$" Let (s, p) be a solution of $(**)$. We have $3^{um} \cdot s_0 = 2^{p_0} + v$ and also $3^{um} \cdot s = 2^p + v$. Thus $2^{p_0} \equiv 2^p \pmod{3^{um}}$, so $2^{p-p_0} \equiv 1 \pmod{3^{um}}$ because $\widehat{2}_{3^{um}}$ is invertible in $\mathbf{Z}_{3^{um}}$ ($(2, 3) = 1$). So $\widehat{2}^{p-p_0}_{3^{um}} = \widehat{1}_{3^{um}}$, hence $2 \cdot 3^{um-1} \mid (p - p_0)$ (from Lemma 3.2 because $2 \cdot 3^{um-1}$ is the order of $\widehat{2}_{3^{um}}$ as we have seen from Theorem 3.2). Thus there exists $t \in \mathbf{N}$ such that $p = p_0 + 2 \cdot 3^{um-1} \cdot t$. On the other hand $s = \frac{2^p+v}{3^{um}}$, so the pair (s, p) is of the requested form.

"$\Longleftarrow$" Let $p = p_0 + 2 \cdot 3^{um-1} \cdot t$ and $s = \frac{2^{p_0 + 2 \cdot 3^{um-1} \cdot t} + v}{3^{um}}$. We shall prove that (s, p) is a solution of $(**)$. We have $2 \cdot 3^{um-1} \cdot t = p - p_0$. It follows that $2^{p-p_0} \equiv 2^{2 \cdot 3^{um-1} \cdot t} \pmod{3^{um}}$, i.e. $2^{p-p_0} \equiv 1 \pmod{3^{um}}$ (because $\varphi(3^{um}) = 2 \cdot 3^{um-1}$, it follows that $2^{2 \cdot 3^{um-1}} \equiv 1 \pmod{3^{um}}$). Hence $2^p \equiv 2^{p_0} \pmod{3^{um}}$, so $2^p + v \equiv 2^{p_0} + v \equiv 0 \pmod{3^{um}}$. Therefore $3^{um} \mid (2^p + v)$, so there exists $s' \in \mathbf{N}$ such that $3^{um} \cdot s' = 2^p + v$. If we prove that $s' = s$, the theorem is proven.

We obtain that $s' = \frac{2^p + v}{3^{um}} = \frac{2^{p_0 + 2 \cdot 3^{um-1} \cdot t} + v}{3^{um}} = s$. Hence $s' = s$, so $3^{um} \cdot s = 2^p + v$, i.e. (s, p) is a solution for $(**)$.

■

Now, we shall apply Theorem 3.3 to some particular values of u and m which are interesting in solving our equations. We try to find numbers p_0 which satisfies $0 \le p_0 < 2 \cdot 3^{um-1}$. This can be done by a simple program. After we find (s_0, p_0), we know all the solutions as stated in Theorem 3.3, b).

Let us denote by $p_0^{(m)}$ the number p_0 which corresponds to a given m. In the following, we describe a relation between $p_0^{(m)}$ and $p_0^{(m+1)}$.

Lemma 3.3 *For equations of type $(**)$, there exists k, $0 \le k < 3^u$, such that $p_0^{(m+1)} - p_0^{(m)} = 2 \cdot 3^{um-1} \cdot k$.*

Proof Let $(s_0^{(m+1)}, p_0^{(m+1)})$ be a solution of the equation $3^{u(m+1)} \cdot s = 2^p + v$. Therefore $(3^u \cdot s_0, p_0^{(m+1)})$ is a solution of the equation $3^{um} \cdot s = 2^p + v$. From Theorem 3.3, b), we infer that there exists $k \in \mathbf{N}$ such that

$$p_0^{(m+1)} = p_0^{(m)} + 2 \cdot 3^{um-1} \cdot k.$$

Because $p_0^{(m+1)} < 2 \cdot 3^{u(m+1)-1}$, it follows that $k < 3^u$.

■

This means we can find $p_0^{(m)}$ using a linear algorithm of the time complexity $\mathcal{O}(m \cdot 3^u) = \mathcal{O}(m)$. According to Theorem 3.3, we would have had an exponential algorithm (we test every number p between 0 and $2 \cdot 3^{um-1} - 1$). By using Lemma 3.3, we can easily implement a program which computes for given values of u, v and for every m, the value of $p_0^{(m)}$. Using this program, we found the form of the solutions for $(u, v) \in \{(2, 5), (7, 17)\}$. We list the form of this solutions below.

Let us consider the equation $3^{2m} \cdot s = 2^p + 5$. We want to find its solutions $(m, s, p) \in \mathbf{N}^3$. We take in Theorem 3.3 $u = 2$ and $v = 5$.

For $m = 0$ we have the solutions $(0, 2^t + 5, t)$, $\forall\ t \in \mathbf{N}$.

For $m = 1$, we find $s_0 = 1$ and $p_0 = 2$ $(0 \le 2 < 2 \cdot 3^{2m-1} = 2 \cdot 3)$. We have the solutions

$$(1, \frac{2^{2 \cdot 3 \cdot t + 2} + 5}{3^2}, 2 \cdot 3 \cdot t + 2),\ \forall\ t \in \mathbf{N}$$

and these are **all** the solutions with $m = 1$.

For $m = 2$, all the solutions are

$$(2, \frac{2^{2\cdot 3^3\cdot t+50}+5}{3^4}, 2\cdot 3^3\cdot t+50),\ \forall\ t\in \mathbf{N}$$

For $m = 3$, all the solutions are

$$(3, \frac{2^{2\cdot 3^5\cdot t+266}+5}{3^6}, 2\cdot 3^5\cdot t+266),\ \forall\ t\in \mathbf{N}$$

For $m = 4$, all the solutions are

$$(4, \frac{2^{2\cdot 3^7\cdot t+266}+5}{3^8}, 2\cdot 3^7\cdot t+266),\ \forall\ t\in \mathbf{N}$$

Let us consider the equation $3^{7m}\cdot s = 2^p + 17$. We take in Theorem 3.3 $u = 7$ and $v = 17$.

For $m = 0$ we have all the solutions $(0, 2^t + 17, t),\ \forall\ t \in \mathbf{N}$.

For $m = 1$, all the solutions are

$$(1, \frac{2^{2\cdot 3^6\cdot t+762}+17}{3^7}, 2\cdot 3^6\cdot t+762),\ \forall\ t\in \mathbf{N}$$

For $m = 2$, all the solutions are

$$(2, \frac{2^{2\cdot 3^{13}\cdot t+1116132}+17}{3^{14}}, 2\cdot 3^{13}\cdot t+1116132),\ \forall\ t\in \mathbf{N}$$

For $(u, v) = (1, 1)$, we find all the solutions by proof, i.e. let us study the equation $3^m\cdot s = 2^p + 1$. In fact, this case has been treated in Theorem 4.2 ([AnM98]). We shall find the solutions $(m, s, p) \in \mathbf{N}^3$. We take $u = 1$ and $v = 1$ in Theorem 3.3. We shall prove that $p_0^{(m)} = 3^{m-1}$. From Lemma 2.4, we have $3^m \mid (2^{3^{m-1}} + 1)$. According to Theorem 3.3, a), because $0 \le 3^{m-1} < 2\cdot 3^{m-1}$ and the uniqueness of p_0 in that range, it follows that $p_0^{(m)} = 3^{m-1}$. Thus, all the solutions of the equation are:

$$(m, \frac{2^{3^{m-1}(2t+1)}+1}{3^m}, 3^{m-1}(2t+1)),\ \forall\ t\in \mathbf{N},\ \forall\ m\ge 1$$

For $m = 0$, we have all the solutions $(0, 2^t + 1, t),\ \forall\ t \in \mathbf{N}$.

Using the above results, we can deduce other subclasses of natural numbers for which Collatz's conjecture holds.

Theorem 3.4 *For all $t\in\mathbf{N}$, $s\in\mathbf{N}$, the following identities hold:*

(i) $\mathbf{f}_C{}^{(5k+s+2\cdot 3\cdot t+2)}\left(2^s\cdot\left[2^{3k}\cdot\frac{2^{2\cdot 3\cdot t+2}+5}{3^{2k}}-5\right]\right) = 1,\ \forall\ k\in\{0,1\}$;

(ii) $\mathbf{f}_{\mathcal{C}}^{(5k+s+2\cdot 3^3\cdot t+50)}\left(2^s\cdot\left[2^{3k}\cdot\frac{2^{2\cdot 2^3\cdot t+50}+5}{3^{2k}}-5\right]\right)=1,\ \forall\ k\in\{0,1,2\}$;

(iii) $\mathbf{f}_{\mathcal{C}}^{(5k+s+2\cdot 3^5\cdot t+266)}\left(2^s\cdot\left[2^{3k}\cdot\frac{2^{2\cdot 3^5\cdot t+266}+5}{3^{2k}}-5\right]\right)=1,\ \forall\ k\in\{0,1,2,3\}$;

(iv) $\mathbf{f}_{\mathcal{C}}^{(5k+s+2\cdot 3^7\cdot t+266)}\left(2^s\cdot\left[2^{3k}\cdot\frac{2^{2\cdot 3^7\cdot t+266}+5}{3^{2k}}-5\right]\right)=1,\ \forall\ k\in\{0,1,2,3,4\}$;

(v) $\mathbf{f}_{\mathcal{C}}^{(18k+s+2\cdot 3^6\cdot t+762)}\left(2^s\cdot\left[2^{11k}\cdot\frac{2^{2\cdot 3^6\cdot t+762}+17}{3^{7k}}-17\right]\right)=1,\ \forall\ k\in\{0,1\}$;

(vi) $\mathbf{f}_{\mathcal{C}}^{(18k+s+2\cdot 3^{13}\cdot t+1116132)}\left(2^s\cdot\left[2^{11k}\cdot\frac{2^{2\cdot 3^{13}\cdot t+1116132}+17}{3^{7k}}-17\right]\right)=1,$
$\forall\ k\in\{0,1,2\}$.

Proof

(i) Using a procedure similar to the proof of Lemma 3.1, we can immediately obtain (by induction on k) the identity:

$$\mathbf{f}_{\mathcal{C}}^{(5k)}\left(3^{2(m-k)}\cdot 2^{3k}\cdot s-5\right)=3^{2m}\cdot s-5,\ \forall\ k\in\{0,1,...,m\}.$$

But, we saw above that the equation $3^{2m}\cdot s-5=2^p$ has the solutions (m,s,p):

$$\left(1,\frac{2^{2\cdot 3\cdot t+2}+5}{3^2},2\cdot 3\cdot t+2\right)$$

According to the fact that $\mathbf{f}_{\mathcal{C}}^{(p)}(2^p)=1$ it follows that

$$\mathbf{f}_{\mathcal{C}}^{(5k+2\cdot 3\cdot t+2)}\left(2^{3k}\cdot\frac{2^{2\cdot 3\cdot t+2}+5}{3^{2k}}-5\right)=1,\ \forall\ k\in\{0,1\}.$$

Now, applying Theorem 1.1 (ii), we can multiply by 2 and thus we immediately obtain the requested identity.

The cases (ii)-(iv) are very similar. For the cases (v), (vi) we can use the identity (which can be proved by induction on k):

$$\mathbf{f}_{\mathcal{C}}^{(18k)}\left(3^{7(m-k)}\cdot 2^{11k}\cdot s-17\right)=3^{7m}\cdot s-17,\ \forall\ k\in\{0,1,...,m\}.$$

■

4 Relations between "even" and "odd" branches

Next, we shall present an interesting result related to the number of "even" and "odd" branches in the Collatz's tree.

We say that $n\to\frac{n}{2}$ is an operation of type 0 and $n\to 3n+1$ of type 1. On a certain number, we cannot perform two consecutive operations of type 1. So, we perform k_0 operations of type 0, one of type 1, k_1 of type 0, one of type 1 and so on.

Theorem 4.1 *Let $a_1, ..., a_t \in \{1,2\}$, $t \in \mathbf{N}_+$ such that there is no $i \in \{1,2,...,t\}$, $a_i = a_{i+1} = 1$. Then, there exists $x \in \mathbf{N}$ such that we can perform on x, the operations of type a_1, ..., a_t in this order.*

Proof Let us suppose that we have n operations of type 1 in this finite sequence. Then there exist k_0, k_1, ..., k_n such that k_0, $k_n \geq 0$ and k_1, ..., $k_{n-1} \geq 1$ and

$$a_{k_0+1} = a_{k_0+k_1+2} = ... = a_{k_0+...+k_{n-1}+n} = 1$$

$$a_i = 0,\ \forall\ i \notin \{k_0+1, k_0+k_1+2, ..., k_0+...+k_{n-1}+n\}$$

because $(2,3) = 1$ it follows that $\widehat{2}_{3^{n-1}} \in \mathbf{U}(\mathbf{Z}_{3^{n-1}})$, i.e. there exists $y \in \mathbf{N}$ for which
$z = 2^k \cdot \left(2^{k_1+...k_n} \cdot y - 3^0 \cdot 2^{k_1+...+k_{n-1}} - 3^1 \cdot 2^{k_1+...k_{n-2}} - ... - 3^{n-2} \cdot 2^{k_1} - 3^{n-1}\right)$ $\equiv 0 \pmod{3^n}$ holds. We take $x = \frac{z}{3^n}$ and we state that this satisfies the requirements. Performing k_0 operations of type 0 on x, we obtain

$$\frac{2^{k_1+...+k_{n-1}} \cdot y - 3^0 \cdot 2^{k_1+...+k_{n-2}} - ... - 3^{n-2} \cdot 2^{k_1} - 3^{n-1}}{3^n}$$

which is an odd number since $k_1 \geq 1$. Performing one operation of type 1 on this number we get

$$\frac{2^{k_1}\left(2^{k_2+...+k_{n-1}} \cdot y - 3^0 \cdot 2^{k_2+...+k_{n-2}} - ... - 3^{n-2}\right)}{3^{n-1}}$$

Continuing in this way, after performing operations of type $a_1, ..., a_t$ in this order on x, we obtain y $(t = k_0 + ... + k_n + n)$.

∎

Let us denote by S_i the set of numbers from Collatz's tree for which we have exactly i operations of type 1 until we reach 1. In [ShW92] it is proved that S_i is infinite. We shall prove this too, as a consequence of Theorem 4.1.

Theorem 4.2 *$card(S_i) = \infty$ (S_i is infinite).*

Proof In Theorem 3.1 we have seen that $\mathbf{f}_C{}^{(2i)}(2^i \cdot s - 1) = 3^i \cdot s - 1$. According to Theorem 3.3, we infer that the equation $3^m \cdot s - 1 = 2^p$ has infinitely many solutions with $m = i$. Because for different s, and fixed i the numbers $2^i \cdot s - 1$ are pairwise distinct, we deduce that $2^i \cdot s - 1 \in S_i$, for any s such that there exists p with $3^i \cdot s - 1 = 2^p$. Moreover, the numbers $2^k \cdot (2^i \cdot s - 1) \in S_i$, $\forall\ k \geq 0$. Thus S_i has infinitely many elements.

∎

For all $n \in \mathbf{N}$ and n from Collatz's tree we denote by a_n the number of steps of type 1, and by b_n the number of steps of type 0 until we reach 1. We try to estimate the ratio $\frac{a_n}{b_n}$. It can be easily seen that $\frac{a_n}{b_n} \leq 1$.

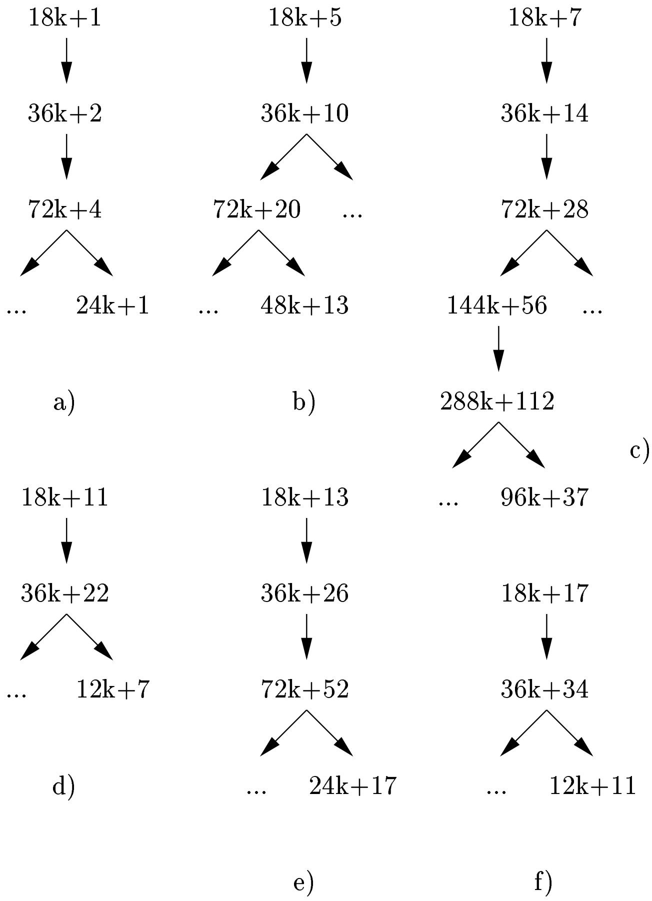
Lemma 4.1 *For every $n \in \mathbf{N} - \{0\}$, with $(n,3) = 1$, there exists $u \in \{1,2,3,4,5,6\}$ such that $2^u \cdot n \equiv 4 \pmod{18}$.*

Proof Because $2^{6t} \equiv 1 \pmod 9$, $2^{6t+1} \equiv 2 \pmod 9$, $2^{6t+2} \equiv 4 \pmod 9$, $2^{6t+3} \equiv 8 \pmod 9$, $2^{6t+4} \equiv 7 \pmod 9$, $2^{6t+5} \equiv 5 \pmod 9$, where $t \in \mathbf{N}$, $(n, 3) = 1$, and $(2, 9) = 1$, we have that there exists a u with $u \in \{1, 2, 3, 4, 5, 6\}$ and $9 \mid (2^{u-1} \cdot n - 2)$. This implies $18 \mid (2^u \cdot n - 4)$.

■

Lemma 4.2 *For every $n \in \mathbf{N} - \{0\}$, with $(n, 6) = 1$, there exist $t, r \in \mathbf{N} - \{0\}$ such that $n = 18t + r$, $r \in \{1, 5, 7, 11, 13, 17\}$, and there exist $m \in \mathbf{N} - \{0\}$, $a = 1$, $b \leq 4$ with $(m, 6) = 1$ and $\mathbf{f}_{\mathcal{C}}{}^{(a+b)}(m) = n$, where a (b) denotes the number of type 1 (0) steps, respectively.*

Proof This follows from the following figure.



■

Lemma 4.3 *There exist infinitely many numbers n, with $(n, 6) = 1$, from Collatz's tree for which we have $\frac{a_n}{b_n} \geq \frac{1}{4}$.*

Proof Let us consider $n_1 = 5$. It is obvious that $(n_1, 6) = 1$, $a_{n_1} = 1$, $b_{n_1} = 4$, and therefore $\frac{a_{n_1}}{b_{n_1}} \geq \frac{1}{4}$. We shall construct n_k inductively for all $k \in \mathbf{N}_+ - \{1\}$.

Let us suppose we have constructed $n_1, ..., n_k$ so that $level(n_i) < level(n_{i+1})$ in Collatz's tree, $(n_i, 6) = 1$ for all $i \in \{1, 2, ..., k\}$, $a_{n_i} = i$, and $\frac{a_{n_i}}{b_{n_i}} \geq \frac{1}{4}$.

Since $(n_k, 6) = 1$ we conclude that there exist $t, r \in \mathbf{N}$ such that $n_k = 18t + r$ with $r \in \{1, 5, 7, 11, 13, 17\}$. From Lemma 4.2 follows that there exists n_k from Collatz's tree such that

- $(n_{k+1}, 6) = 1$,
- $a_{n_{k+1}} = a_{n_k} + 1$,
- $b_{n_{k+1}} \leq b_{n_k} + 4$,
- $level(n_k) < level(n_{k+1}) \leq level(n_k) + 5$.

From the induction hypothesis it is obvious that $a_{n_{k+1}} = a_{n_k} + 1 = k + 1$ and $b_{n_{k+1}} \leq b_{n_k} + 4 \leq 4 \cdot a_{n_k} + 4 = 4 \cdot a_{n_{k+1}}$. Therefore $\frac{a_{n_{k+1}}}{b_{n_{k+1}}} \geq \frac{1}{4}$.

Because of $level(n_1) < level(n_2) < ...$ and Lemma 2.1 follows that the numbers n_1, n_2, ... are pairwise distinct.

■

Remark 4.1 *The numbers $n_1, n_2, \cdots$ belong to the same infinite path in Collatz's tree as can be easily seen from the construction above. Moreover, this infinite path is not ultimately ending in a chain. Let n be any number on this path (starting with n_1). Then there exists $k \in \mathbf{N} - \{0\}$ such that n is between $n-k$ and n_{k+1} on this path. It follows that $a_{n_k} = a_n = k$ and $b_{n_k} < b_n \leq b_{n_k} + 4$. Therefore $\frac{a_n}{b_n} = \frac{k}{b_n} \geq \frac{k}{b_{n_k}+4} \geq \frac{k}{4a_{n_k}+4} = \frac{k}{4(k+1)} = \frac{1}{4} - \frac{1}{4(k+1)} \geq \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$.*

Thus, $\frac{a_n}{b_n} \geq \frac{1}{8}$ holds for all numbers n on this path.

Theorem 4.3 *The limit $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ does not exist.*

Proof We know that $\lim\limits_{i\to\infty} \frac{a_{2^i}}{b_{2^i}} = \lim\limits_{i\to\infty} \frac{0}{i} = 0$. Note that the sequence (2^i) is an infinite path of Collatz's tree but not ultimately a chain since there are infinitely many branching points by Lemma 4.1.

According to Lemma 4.3, there exist infinitely many numbers n such that n is from Collatz's tree and $\frac{a_n}{b_n} \geq \frac{1}{4}$. From these numbers, we can extract $n_{k_1} < n_{k_2} < ....$

If the limit $\lim\limits_{i\to\infty} \frac{a_{n_{k_i}}}{b_{n_{k_i}}}$ exists, this is greater or equal than $\frac{1}{4}$. But $\lim\limits_{i\to\infty} \frac{a_{2^i}}{b_{2^i}} = 0$. Thus the limit $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ doesn't exist. If the limit $\lim\limits_{i\to\infty} \frac{a_{n_{k_i}}}{b_{n_{k_i}}}$ doesn't exist, then the limit $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ doesn't exist either.

■

It is obvious that we cannot perform 2 consecutive steps of type 1 on a number n. Since the last 3 steps are of type 0 we get $a_n < b_n$, or $\frac{a_n}{b_n} < 1$.

In the following we prove a stronger result.

Theorem 4.4 *For all n from Collatz's tree holds* $\frac{n \cdot 3^{a_n}}{2^{b_n}} \leq 1$.

Proof This will be shown by induction on $a_n + b_n$. Note that $a_n + b_n$ ranges over all positive natural numbers since there exists the infinite set $n = b^k$ with $a_n = 0$.

The basis is obvious since to $a_n + b_n = 1$ only corresponds $n = 2$ with $a_n = 0$ and $b_n = 1$.

Assume that the statement holds for all m such that $a_m + b_m = k$, and suppose that n is from Collatz's tree with $a_n + b_n = k + 1$. Let m be the father of n, thus $a_m + b_m = k$. Now, either $m = \frac{n}{2}$ or $m = 3n + 1$.

In the first case we have $a_n = a_m$ and $b_n = b_m + 1$, and therefore we get $\frac{n \cdot 3^{a_n}}{2^{b_n}} = \frac{2m \cdot 3^{a_m}}{2^{b_m+1}} = \frac{m \cdot 3^{a_m}}{2^{b_m}} \leq 1$ (by the induction hypothesis).

In the second case we have $a_n = a_m + 1$ and $b_n = b_m$, and therefore we get $\frac{n \cdot 3^{a_n}}{2^{b_n}} = \frac{3n \cdot 3^{a_m}}{2^{b_m}} = \frac{(m-1)3^{a_m}}{2^{b_m}} < \frac{m \cdot 3^{a_m}}{2^{b_m}} \leq 1$ (***) (by the induction hypothesis).

Therefore, in both cases we have $\frac{n \cdot 3^{a_n}}{2^{b_n}} \leq 1$.

■

Remark 4.2 *The equality* $\frac{n \cdot 3^{a_n}}{2^{b_n}} = 1$ *only holds for n of the form* $n = 2^k$. *This can be seen easily from (***).*

Theorem 4.5 *For all n from Collatz's tree holds* $\frac{a_n}{b_n} < \frac{log2}{log3}$.

Proof From Theorem 4.4 we have $\frac{n \cdot 3^{a_n}}{2^{b_n}} \leq 1$, and therefore $n \cdot 3^{a_n} \leq 2^{b_n}$ which implies $3^{a_n} < 2^{b_n}$ being equivalent to $a_n \cdot log3 < b_n \cdot log2$. Therefore $\frac{a_n}{b_n} < \frac{log2}{log3} < 1$.

■

Remark 4.3 *The last two theorems may be generalized to the case that the root of the tree is not* 1 *but an arbitrary number* $q \in \mathbf{N}$ *with* $q \not\equiv 4 \pmod 6$. *With the same meaning for* a_n *and* b_n *to reach q hold:* $\frac{n3^{a_n}}{2^{b_n}} \leq q$ *with* $\frac{n3^{a_n}}{2^{b_n}} = q$ *only if* $n = q2^k$, *and* $\frac{a_n}{b_n} < \frac{log2+logq}{log3}$.

The following result establishes in which cases some similarities in Collatz's tree may occur.

Theorem 4.6 *If* n_1 *and* n_2 *belong to Collatz's tree,* $n_1,\ n_2 \geq 4$, *and the difference* $n_1 - n_2 = k_1 \cdot 3^{k_2}$, *where* k_1 *is even, then* $A_{2k_2}(n_1)$ *and* $A_{2k_2}(n_2)$ *have the same structure (i.e. are isomorphic subtrees).*

Proof For $k_2 = 0$ the statement is obvious.

We shall prove the general statement by induction on k_2. We suppose the statement true for $k_2 - 1$. For $k_2 \geq 1$ we have (because k_1 is even) $n_1 - n_2 \equiv 0 \pmod 6$. Let us suppose $n_1 = 6t_1 + r$ and $n_2 = 6t_2 + r$, where $r \in \{0, 1, 2, 3, 4, 5\}$

and t_1, t_2 are natural numbers. In the following figure we display all possible situations:

$$\begin{tikzcd} 6k \arrow[d] \\ 12k \arrow[d] \\ 24k \end{tikzcd}$$

a)

$$\begin{tikzcd} 6k+1 \arrow[d] \\ 12k+2 \arrow[d] \\ 24k+4 \end{tikzcd}$$

b)

$$\begin{tikzcd} & 6k+2 \arrow[d] & \\ & 12k+4 \arrow[dl] \arrow[dr] & \\ 24k+8 & & 4k+1 \end{tikzcd}$$

c)

$$\begin{tikzcd} 6k+3 \arrow[d] \\ 12k+6 \arrow[d] \\ 24k+12 \end{tikzcd}$$

d)

$$\begin{tikzcd} & 6k+4 \arrow[dl] \arrow[dr] & \\ 12k+8 \arrow[d] & & 2k+1 \arrow[d] \\ 24k+16 & & 4k+2 \end{tikzcd}$$

e)

$$\begin{tikzcd} & 6k+5 \arrow[d] & \\ & 12k+10 \arrow[dl] \arrow[dr] & \\ 24k+20 & & 4k+3 \end{tikzcd}$$

f)

If $r = 0$ we are in situation a). The first two levels of $A_{2k_2}(n_1)$ and $A_{2k_2}(n_2)$ have the same structure. Moreover

$$24t_1 - 24t_2 = 24(t_1 - t_2) = 4(n_1 - n_2) = 4k_1 \cdot 3^{k_2} = 12k_1 \cdot 3^{k_2 - 1}$$

Applying the induction hypothesis, we infer that $A_{2k_2}(n_1)$ and $A_{2k_2}(n_2)$ have the same structure.

If $r = 5$ we are in situation f). The first two levels of $A_{2k_2}(n_1)$ and $A_{2k_2}(n_2)$ have the same structure. Moreover

$$24t_1 + 20 - (24t_2 + 20) = 24(t_1 - t_2) = 4(n_1 - n_2) = 4k_1 \cdot 3^{k_2} = 12k_1 \cdot 3^{k_2 - 1} \text{ and}$$

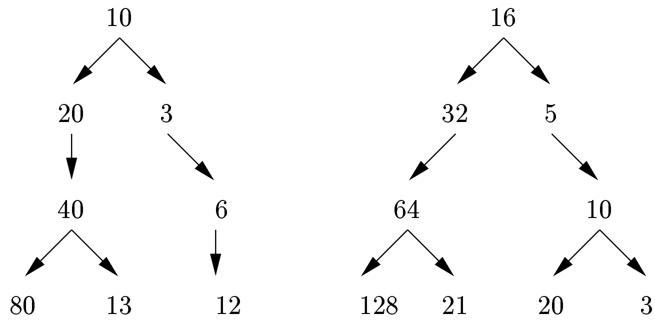
$$4t_1 + 3 - (4t_2 + 3) = 4(t_1 - t_2) = 4 \cdot \frac{n_1 - n_2}{6} = 2k_1 \cdot 3^{k_2 - 1}$$

Applying the induction hypothesis, we infer that $A_{2k_2}(n_1)$ and $A_{2k_2}(n_2)$ have the same structure.

Situations b), c), d), e) can be handled in a similar way. Having proved the induction step, the proof is complete.

$\blacksquare$

Remark 4.4 *Related to Theorem 4.6, we cannot state the theorem for subtrees* $A_{2k_2+1}(n_1)$ *and* $A_{2k_2+1}(n_2)$. *As a counterexample,* $A_3(10)$ *and* $A_3(16)$ *have different structures on level* 3.



5 Conclusions

For the sake of presentation, we can give a new reformulation (the fourth one) of Collatz's conjecture using a type 0 grammar. Let

$$G = (\{S, A, B, C, D\}, \{\#, a\}, S, P)$$

be a type 0 grammar for which the set of productions P is given by:

1. $S \to \# a \#$	2. $S \to \# a a \#$
3. $S \to \# a a a a \#$	4. $S \to \# A A A A A A A A \#$
5. $\# A \to \# A B$	6. $B A \to A A B$
7. $B \# \to A \#$	8. $\# A A \to \# A A C$
9. $C A A \to A A C$	10. $A C \# \to D \#$
11. $A A A D \to D A$	12. $\# D \to \#$
13. $A \to a$	

The grammar G is not monotonic because of the productions 10, 11, 12. It is almost obvious that G (which we called **Collatz's grammar**) simulates Collatz's tree $A(8)$, i.e.:

$$\# a^n \# \in L(G),\ n \in \mathbf{N} \iff n \text{ is a label of a node from } A(8)$$

To the numbers $1, 2, 4$, which are on the levels $0, 1, 2$ in $A(1)$, correspond the terminal words $\# a \#$, $\# a a \#$, $\# a a a a \#$ (obtained from the productions $1, 2, 3$). The label of the root of $A(8)$ is described in the fourth production of G. The transformation $y \to 2 \cdot y$ is simulated by the productions $5, 6, 7$ and the transformation $y \to \frac{y-1}{3}$, when $y \equiv 4 \pmod 6$ is simulated by the productions $8, 9, 10, 11, 12$. The productions $8, 9, 10$ check if the number of A symbols is even, and the productions $10, 11, 12$ check if the number of A symbols is $1 \pmod 3$. Therefore, in the affirmative case, the number of A symbols is $4 \pmod 6$. Finally, using production 13, we obtain the corresponding terminal word.

For instance, we can easily prove that

$$(i) \qquad \# A^n \# \overset{n+1,*}{\underset{G}{\Longrightarrow}} \# A^{2n} \# \text{ using the productions 5,6,7}$$

and

$$(ii) \qquad \# A^{6k+4} \# \overset{5k+4,*}{\underset{G}{\Longrightarrow}} \# A^{2k+1} \# \text{ using the productions 8,9,10,11,12}$$

where $\overset{n+1,*}{\underset{G}{\Longrightarrow}}$ means that we have applied $n+1$ derivations in G.

One of the property of this grammar is:

$$\text{if there exists a derivation } S \overset{k,*}{\underset{G}{\Longrightarrow}} \# a^n \#, \text{ then } k \text{ is unique.}$$

If we don't want to have this property, we can just simply replace the productions $1, 2, 3, 4$ by $S \to \# A \#$ (i.e. we obtain an equivalent phrase structure grammar with only 10 productions). As a conclusion, maybe this reformulation may lead to further investigations.

In our paper, we find some new infinite subclasses of natural numbers for which Collatz's conjecture is true. We discuss the chain subtrees and their properties. As a conjecture very similar with Collatz's conjecture we present the problem (C) (section 3). Another open problem is to estimate the relation between the number of "even" steps and "odd" steps in the Collatz's tree. A result from [ShW92] has been proved as a consequence of Theorem 4.1.

Acknowledgements

We thank to Prof. Dr. Matthias Jantzen and Research Assistent Olaf Kummer (Fachbereich Informatik, Universität Hamburg) for their very useful remarks, suggestions and comments which improved the paper.

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