A Note on Hack's Conjecture, Parikh Images of Matrix Languages and Multiset Grammars

Georg Zetzsche

University of Hamburg, Department of Computer Science
georg.zetzsche@informatik.uni-hamburg.de

Abstract

It is shown that Hack's Conjecture on Petri nets implies that for every language generated by a matrix grammar (without appearance checking), there is a non-erasing matrix grammar generating a language of the same Parikh image. Is is also shown that in this case, the classes of multiset languages generated by arbitrary and monotone multiset grammars coincide.

Zusammenfassung

Es wird gezeigt, dass die Hack'sche Vermutung über Petri-Netze impliziert, dass es für jede von einer Matrix-Grammatik (ohne Vorkommenstest) erzeugte Sprache eine nicht-löschende solche Grammatik gibt, die eine Sprache vom gleichen Parikh-Bild erzeugt. Es wird auch gezeigt, dass in diesem Fall die Klassen der Multiset-Sprachen, die von beliebigen bzw. monotonen Multiset-Grammatiken erzeugt werden, übereinstimmen.

1 Introduction

For definitions and notation, we refer the reader to [Zet09].

In [Hac76, p. 172], Michel Hack conjectured that the reachability problem for Petri nets is decidable. The conjecture also states that this is due to the fact that for every Petri net of size[1] $y \in \mathbb{N}$, a constant $K_y \in \mathbb{N}$ can be determined such that for any marking μ, the zero marking is reachable from μ iff it can be reached by a firing sequence of length less than $K_y \cdot \|\mu\|$.

[1]Hack defines the size of a Petri net to be its total number of arcs (see [Hac76, p. 170]).

Although it is known that the reachability problem for P/T nets is decidable (see [May81, May84]), this stronger conjecture remains open and has interesting implications. In this paper, it is shown that if Hack's Conjecture holds, the classes **MAT** and $\mathbf{MAT}^\lambda$ coincide with respect to their Parikh image. That is, for every matrix grammar G (without appearance checking), there is a non-erasing matrix grammar G' such that $\Psi(L(G')) = \Psi(L(G))$. In other words, $\Psi(\mathbf{MAT}^\lambda) \subseteq \Psi(\mathbf{MAT})$ and therefore $\Psi(\mathbf{MAT}) = \Psi(\mathbf{MAT}^\lambda)$. Note that whether the classes **MAT** and $\mathbf{MAT}^\lambda$ are equal is an open question in the theory of regulated rewriting.

The result $\Psi(\mathbf{MAT}^\lambda) = \Psi(\mathbf{MAT})$ also means that the multiset language classes **mARB** and **mMON** coincide. **mARB** (**mMON**) is the class of multiset languages generated by arbitrary (monotone) multiset grammars (see [KMVP01] for details on multiset grammars). This result is due to the fact that $\mathbf{mARB} = \Psi(\mathbf{MAT}^\lambda)$ and $\mathbf{mMON} = \Psi(\mathbf{MAT})$.

For more information on Hack's conjecture, see [Gra79].

2 Petri net languages

A language of the form L_φ in the next lemma will be needed in one of the later proofs.

Lemma 1. *For any monoid-homomorphism $\varphi : \Sigma^* \to \mathbb{Z}$, the language $L_\varphi = \{w \in \Sigma^* \mid \varphi(w) \geq 0\}$ is in $\mathcal{L}_0$.*

Proof. We construct a λ-free Petri net $N = (\Sigma, P, T, \partial_0, \partial_1, \sigma, \mu_0, F)$. Let $M := \max\{|\varphi(a)| \mid a \in \Sigma\}$, $\Sigma_- := \{a \in \Sigma \mid \varphi(a) < 0\}$, and $\Sigma_+ := \{a \in \Sigma \mid \varphi(a) \geq 0\}$. The set of transitions is

$$\begin{aligned} T := \ & \{t^+_{a,r,s} \mid a \in \Sigma_+,\ 0 \leq r \leq M,\ 0 \leq s \leq \varphi(a)\} \\ & \cup \{t^-_{a,r} \mid a \in \Sigma_-,\ 0 \leq r \leq M\} \end{aligned}$$

and the set of places is $P = \{p_+, p_-\}$. For any $a \in \Sigma_+$, $0 \leq r \leq M$, $0 \leq s \leq \varphi(a)$, $\sigma(t^+_{a,r,s}) := a$ and for any $a \in \Sigma_-$, $0 \leq r \leq M$, $\sigma(t^-_{a,r}) := a$. The initial marking is $\mu_0 := \mathbf{0}$ and the final markings are

$$F := \{r \cdot (p_+ + p_-) \mid 0 \leq r \leq M\}.$$

The net will work as follows. For $\mu \in P^\oplus$, let $\psi(\mu) := \mu(p_+) - \mu(p_-)$. If $\varphi(a) \geq 0$, the firing of a a-labeled transition $t^+_{a,r,s}$ increases the image of the marking under ψ by a value between 0 and $\varphi(a)$. Furthermore, it subtracts r from both p_+ and p_-. The latter does not change the image of the marking under ψ but makes sure that the markings can be kept small.

If $\varphi(a) < 0$, then the transitions $t^-_{a,r}$ add $\varphi(a)$ to the image of the marking under ψ. Besides, they subtract a certain value from both places. The pre- and post-multisets are as follows. For $a \in \Sigma$, $0 \leq r \leq M$, and, in case $a \in \Sigma_+$, $0 \leq s \leq \varphi(a)$, let

$$\begin{array}{lll} \partial_0(t^+_{a,r,s}) := r \cdot (p_- + p_+), & \partial_1(t^+_{a,r,s}) := s \cdot p_+, & \text{for } a \in \Sigma_+, \\ \partial_0(t^-_{a,r}) := r \cdot (p_- + p_+), & \partial_1(t^-_{a,r}) := (-\varphi(a)) \cdot p_-, & \text{for } a \in \Sigma_-. \end{array}$$

Let $\mu \in P^\oplus$ be reachable by a sequence labeled with $w \in \Sigma^*$. By induction on the length of $w \in \Sigma^*$, one can see that $\psi(\mu) \leq \varphi(w)$. The fact that $\psi(\mu) = 0$ for every $\mu \in F$ now shows that $L(N) \subseteq L_\varphi$.

On the other hand, the following fact is also clear by induction on the length of w. For every $w \in \Sigma^*$, there is a firing sequence s, $\sigma(s) = w$, that leads to a marking μ such that $\mu(p_+), \mu(p_-) \leq M$ and

- if $\varphi(w) \geq 0$, then $\psi(\mu) = 0$,
- if $\varphi(w) < 0$, then $\psi(\mu) = \varphi(w)$.

Therefore, we have $L_\varphi \subseteq L(N)$. □

3 Hack's Conjecture

The equality $\Psi(\mathbf{MAT}) = \Psi(\mathbf{MAT}^\lambda)$ can already be deduced from a slightly weaker version of Hack's Conjecture, which will be stated here.

Conjecture 2 (Hack's Conjecture)**.** *For every Petri net N, there is a constant $K \in \mathbb{N}$ such that for any marking μ, the empty marking is reachable from μ iff it can be reached by a firing sequence of length less than $K \cdot \|\mu\|$.*

The difference between this version and Hack's version is that in the latter, the computability of the constant K_y from the size y is also stated. Furthermore, Conjecture 2 only requires the constant K to depend on N, whereas Hack's original conjecture states that the constant only depends on the size of N. This, however, is not a weaker requirement, since, up to initial and final markings, there are only finitely many Petri nets of a certain size. Therefore, if there is such a constant for every Petri net, then there is a constant for every given size.

We will need the following result from the article [Zet09]. It states that applying linear erasing homomorphisms to languages generated by λ-free Petri nets yields languages that can be generated by non-erasing matrix grammars.

Theorem 3 ([Zet09])**.** $\mathcal{H}^{\mathrm{lin}}(\mathcal{L}_0) \subseteq \mathbf{MAT}$.

The next lemma states that arbitrary matrix languages and Petri net languages have the same Parikh image. For a proof, see [HJ94].

Lemma 4 ([HJ94])**.** $\Psi(\mathbf{MAT}^\lambda) = \Psi(\mathcal{L}_0^\lambda)$.

The following is the key lemma in our proof, since it describes the consequences of Hack's Conjecture in terms of Petri net languages.

Lemma 5. *Suppose Hack's Conjecture holds. Let $L \subseteq \Sigma^*$ be in $\mathcal{L}_0$ and $x \in \Sigma$ be a symbol. Then there is a $k \in \mathbb{N}$ such that for any word $w \in L \setminus \{x\}^*$, there is a $v \in L$ with $\Psi(\delta_x(v)) = \Psi(\delta_x(w))$ and $|v| \leq k \cdot |\delta_x(v)|$.*

Proof. Let $N = (\Sigma, P, T, \partial_0, \partial_1, \sigma, \mu_0, F)$ be a λ-free Petri net such that $L(N) = L$ and let K be the constant from Hack's Conjecture. From N, we construct a Petri net $N' = (\Sigma', P', T', \partial_0', \partial_1', \sigma', \mu_0', F')$, to which we will apply Hack's Conjecture. Let $\Sigma' := \Sigma \setminus \{x\}$ and let p_a for every $a \in \Sigma'$ and r be new places. Furthermore, let t_μ be a new transition for every $\mu \in F$, and let $\mu_0' := \mu_0 + r$. The new set of places is then $P' := P \cup \{p_a \mid a \in \Sigma'\} \cup \{r\}$ and the new set of transitions is $T' := T \cup \{t_\mu \mid \mu \in F\}$. For any $t \in T$ and any $\mu \in F$, let

$$\partial_0'(t) := \begin{cases} r + \partial_0(t) + p_{\sigma(t)} & \text{if } \sigma(t) \neq x, \\ r + \partial_0(t) & \text{otherwise,} \end{cases} \qquad \partial_1'(t) := r + \partial_1(t),$$

$$\begin{array}{ll} \partial_0'(t_\mu) := r + \mu, & \partial_1'(t_\mu) := \mathbf{0}, \\ \sigma'(t) := \sigma(t), & \sigma'(t_\mu) := \lambda. \end{array}$$

The embedding morphism $\iota : \Sigma'^\oplus \to P'^\oplus$ is defined by $\iota(a) := p_a$ for $a \in \Sigma'$. Since $w \notin \{x\}^*$, we have $|\delta_x(w)| \geq 1$. Since $w \in L$, there is a firing sequence s in N' that leads from $\nu_w := \mu_0' + \iota(\Psi(\delta_x(w)))$ to $\mathbf{0}$. It follows from the hypothesis that there is also a firing sequence s' leading from ν_w to $\mathbf{0}$ such that $|s'| < K \cdot \|\nu_w\|$. With $v := \sigma'(s')$, we have $\Psi(\delta_x(v)) = \Psi(\delta_x(w))$ and thus

$$\begin{aligned} |v| &\leq |s'| < K \cdot \|\nu_w\| = K \cdot (\|\mu_0'\| + |\delta_x(w)|) = K \cdot (\|\mu_0'\| + |\delta_x(v)|) \\ &\leq K \cdot (\|\mu_0'\| \cdot |\delta_x(v)| + |\delta_x(v)|) = K(\|\mu_0'\| + 1) \cdot |\delta_x(v)|. \end{aligned}$$

Therefore, $k := K(\|\mu_0'\| + 1) = K(\|\mu_0\| + 2)$ meets our requirements. $\square$

We will now use the last lemma to infer an inclusion of multiset language classes.

Lemma 6. *If Hack's Conjecture holds, then* $\Psi(\mathcal{L}_0^\lambda) \subseteq \Psi(\mathcal{H}^{\text{lin}}(\mathcal{L}_0))$.

Proof. Let $L \subseteq \Sigma^*$ be in $\mathcal{L}_0^\lambda$ and let $x \in \Sigma$ be a symbol that does not occur in L. Without loss of generality, we can assume that $\lambda \notin L$. Then write $L = \delta_x(M)$ for some $M \subseteq \Sigma^*$ in $\mathcal{L}_0$. Note that $M \cap \{x\}^* = \emptyset$.

For M, Lemma 5 yields a k with the property stated there. Let $R_{\Sigma,k}(x)$ be defined by

$$R_{\Sigma,k}(x) := \{w \in \Sigma^* \mid |w| \leq k \cdot |\delta_x(w)|\},$$

which is in $\mathcal{L}_0$ according to Lemma 1. Since $\mathcal{L}_0$ is closed under intersection, the language $M' = M \cap R_{\Sigma,k}(x)$ is still in $\mathcal{L}_0$. The property from Lemma 5 implies $\Psi(\delta_x(M)) = \Psi(\delta_x(M'))$. The homomorphism δ_x is linear erasing on $M' \subseteq R_{\Sigma,k}(x)$. Therefore, $\Psi(L) = \Psi(\delta_x(M)) = \Psi(\delta_x(M'))$ is in $\Psi(\mathcal{H}^{\text{lin}}(\mathcal{L}_0))$. □

We are now ready to prove the main result.

Theorem 7. *If Hack's Conjecture holds, then* $\Psi(\mathbf{MAT}) = \Psi(\mathbf{MAT}^\lambda)$.

Proof. $\Psi(\mathbf{MAT}) \subseteq \Psi(\mathbf{MAT}^\lambda)$ follows directly from the definition. Lemma 4, Lemma 6, and Theorem 3 imply

$$\Psi(\mathbf{MAT}^\lambda) = \Psi(\mathcal{L}_0^\lambda) \subseteq \Psi(\mathcal{H}^{\text{lin}}(\mathcal{L}_0)) \subseteq \Psi(\mathbf{MAT}). \qquad \square$$

It is a well-known fact that $\Psi(\mathbf{MAT}^\lambda) = \mathbf{mARB}$ and $\Psi(\mathbf{MAT}) = \mathbf{mMON}$ (see [KMVP01, Theorem 1]). This yields the following corollary.

Corollary 8. *If Hack's Conjecture holds, then* $\mathbf{mARB} = \mathbf{mMON}$.

References

[Gra79] Jan Grabowski. On Hack's Conjecture Concerning Reachability in Petri Nets. *Elektronische Informationsverarbeitung und Kybernetik*, 15(7):339–354, 1979.

[Hac76] M. Hack. Decidability Questions for Petri Nets. Technical Report 161, Massachusetts Institute of Technology, Laboratory of Computer Science, June 1976.

[HJ94] Dirk Hauschildt and Matthias Jantzen. Petri Net Algorithms in the Theory of Matrix Grammars. *Acta Informatica*, (31):719–728, 1994.

[KMVP01] Manfred Kudlek, Carlos Martín-Vide, and Gheorghe Păun. Toward a Formal Macroset Theory. In *Multiset Processing – Mathematical, Computer Science, and Molecular Computing Points of View*, volume 2235, pages 123–133. 2001.

[May81] Ernst W. Mayr. An Algorithm for the General Petri Net Reachability Problem. In *STOC '81: Proceedings of the thirteenth annual ACM symposium on Theory of computing*, pages 238–246, New York, NY, USA, 1981. ACM.

[May84] Ernst W. Mayr. An Algorithm for the General Petri Net Reachability Problem. *SIAM J. Comput.*, 13(3):441–460, 1984.

[Zet09] Georg Zetzsche. Erasing in Petri Net Languages and Matrix Grammars. In Volker Diekert and Dirk Nowotka, editors, *Developments in Language Theory, 13th International Conference, DLT 2009, Stuttgart, Germany, June 30–July 3, 2009. Proceedings*, volume 5583, pages 490–501, 2009.